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# On higher spins and the tensionless limit of String Theory

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## Abstract

We discuss string spectra in the low-tension limit using the BRST formalism, with emphasis on the role of triplets of totally symmetric tensors and spinor-tensors and their generalizations to cases with mixed symmetry and to (A)dS backgrounds. We also present simple compensator forms of the field equations for individual higher-spin gauge fields that display the unconstrained gauge symmetry of a previous non-local construction and reduce upon partial gauge fixing to the (Fang-)Fronsdal equations. For Bose fields we also show how a local Lagrangian formulation with unconstrained gauge symmetry is determined by a previous BRST construction.

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## 1. Introduction

Higher-spin gauge fields are a fascinating topic in Field Theory that still presents a variety of obscure features and open problems. To wit, even the basic formulation of their free dynamics, first proposed long ago by Fronsdal [1] for Bose fields and by Fang and Fronsdal [2] for Fermi fields, was recently shown to result from a partial gauge fixing of Maxwell-like or Einstein-like geometric equations [3, 4] involving the linearized higher-spin curvatures  $\mathcal{R}$  introduced long ago by de Wit and Freedman [5] (see also [6]),

$$\frac{1}{\square^p} \partial \cdot \mathcal{R}^{[p];\alpha_1 \cdots \alpha_{2p+1}} = 0 \quad (1.1)$$

for odd spins  $s = 2p + 1$ , and

$$\frac{1}{\square^{p-1}} \mathcal{R}^{[p];\alpha_1 \cdots \alpha_{2p}} = 0 \quad (1.2)$$

for even spins  $s = 2p$ , where  $[p]$  indicates a  $p$ -fold trace of the curvatures  $\mathcal{R}^{\mu_1 \cdots \mu_s; \nu_1 \cdots \nu_s}$ . Only the two familiar (Maxwell and Einstein) cases of these equations are local, while all others contain *non-local terms* starting at spin  $s = 3$ . Still, their gauge fixing to the local Fronsdal form can be attained at the expense of the trace of the gauge parameter  $\Lambda$ , denoted by  $\Lambda'$  in the following and constrained to vanish in the Fronsdal formulation [4] along with the double trace of the gauge field. Strictly speaking, the non-local geometric equations of [3] apply to totally symmetric tensors, a wide and interesting class of higher-spin gauge fields that does not exhaust all possibilities in more than four dimensions, but tensors of mixed symmetry were recently discussed in these terms in [7]. Therefore, one can go beyond the Fronsdal formulation for general tensor gauge fields, eliminating the need for constraints on the gauge fields themselves or on the gauge parameters. Still, in order to test the role of the unconstrained gauge symmetry in the presence of interactions an equivalent *local* form, obtained combining the basic gauge fields with suitable *compensators*, appears potentially quite useful. In [3] such a formulation was presented for the relatively simple case of a spin-3 field, and one of the results of the present paper is its generalization to symmetric tensors of arbitrary rank.

Thanks primarily to the work of Vasiliev [8, 9] (see also the recent work on higher-dimensional and supersymmetric extensions by Sezgin and Sundell [10]), much is now known about higher-spin interactions, whereas for a long time only negative results, pointing to the extremely subtle nature of these systems, have been available. For instance, an early,

classic result in this context was the Aragone-Deser problem, arising in the presence of gravitational backgrounds with a non-trivial Weyl tensor [11]. Since the gauge invariance of the flat-space Fronsdal Lagrangian rests crucially on the commuting nature of partial derivatives, the extension to a curved space must face the potential emergence of terms proportional to the background Weyl tensor arising from commutators of covariant derivatives, that would jeopardize the gauge symmetry. Surely enough, such commutators are present also for lower spins, but they always combine into Ricci tensors, and for instance supergravity provides a well-known example of this phenomenon [12]. Conformally flat space times, and in particular the familiar and important cases of (anti)de Sitter spaces, collectively denoted by (A)dS in the following, have vanishing Weyl tensors and therefore should allow the consistent propagation of individual higher-spin fields. Indeed, the results of [1, 2] were soon generalized to (A)dS space-times in [13], but as we shall see even these cases present some surprising features. In more general backgrounds, the current understanding is that an infinite number of such fields in mutual interaction is needed to define a consistent dynamics.

The work of Vasiliev [9] (see also [10]) culminates in the proposal of consistent interacting higher-spin equations resulting from the gauging of an infinite-dimensional generalization of the tangent-space Lorentz algebra that underlies Einstein's theory in the vielbein formalism. Vasiliev's construction is also based on the vielbein formalism, whereby the tangent space algebra is enlarged while only ordinary diffeomorphisms are left as manifest symmetries. From this viewpoint, the constraints present in the Fronsdal formulation only involve algebraic conditions relating tangent-space tensors to the Minkowski metric, but it is nonetheless interesting to see whether the extended gauge symmetry of [3, 4] can be accommodated in a suitable formulation, and gaining some understanding of this issue was a main motivation for the present work. In addition, we should stress that the Vasiliev equations, whereas consistent, are intrinsically non-Lagrangian, since they lack additional fields needed in an off-shell formulation, and this is a key open problem in higher-spin dynamics today. Our results will display simple instances of this type of phenomenon, since, for instance, the local compensator equations with unconstrained gauge symmetry we shall meet will also come in two varieties, a simple and compact non-Lagrangian form and a more involved off-shell Lagrangian one.

String Theory, to some extent a more familiar system, includes infinitely many higher-spin massive fields with consistent mutual interactions, and can provide useful hints on their dynamics, if a suitable limit where their masses disappear is explored. This is the low-tension limit, and conversely one can well hope that a better grasp of higher-spin dynamics could help forward our current understanding of String Theory, that is mostly based on its low-spin massless excitations and on their low-energy interactions.

In this respect, the purpose of this paper is thus twofold. On the one hand, we describe the “triplets”, first discussed in 1986 by A. Bengtsson [14] and identified in general in [4], and their generalizations, that make up the full bosonic string spectrum in the tensionless limit, with special emphasis on the relatively simple case of fully symmetric tensors, and show how to extend them to the case of (A)dS backgrounds. These systems comprise a spin- $s$  field  $\varphi$ , a spin- $(s - 1)$  field  $C$  and a spin- $(s - 2)$  field  $D$ , and the corresponding flat-space equations read

$$\begin{aligned} \square \varphi &= \partial C , \\ C &= \partial \cdot \varphi - \partial D , \\ \square D &= \partial \cdot C , \end{aligned} \tag{1.3}$$

where, as will often be the case in the following, tensor indices are left implicit. They propagate a chain of modes of spin  $s, s - 2, \dots, 0$  or  $1$  according to whether  $s$  is even or odd, and were also considered in [15] as a natural arena for the BRST technique [16]. On the other hand, as we shall see, their study is rather rewarding, since they provide a direct route toward the formulation of non-Lagrangian *local* field equations for higher-spin gauge fields. For instance, in flat space this local compensator form of the bosonic equations for a spin- $s$  field  $\varphi$  is simply

$$\begin{aligned} \mathcal{F} &= 3\partial^3 \alpha , \\ \varphi'' &= 4\partial \cdot \alpha + \partial \alpha' , \end{aligned} \tag{1.4}$$

where  $\alpha$  a spin- $(s - 3)$  compensator. This is to be compared with the usual Fronsdal equation

$$\mathcal{F} \equiv \square \varphi - \partial \partial \cdot \varphi + \partial^2 \varphi' = 0 , \tag{1.5}$$

where the gauge field  $\varphi$  is subject to the constraint that its double trace  $\varphi''$  vanish identically. We shall derive these remarkably simple equations for both Bose and Fermi fields, in flat space and in (A)dS backgrounds, that play a crucial role in the Vasiliev equations. For Bose fields, we shall be able to proceed even further, adapting the BRST procedure to the string in the tensionless limit to extend the compensator equations (1.4) to a Lagrangian form in flat space. This result is actually contained in [17] where, however, it was connected to the conventional Fronsdal formulation. Here, on the other hand, we display its natural link with the unconstrained gauge symmetry of the non-local geometric equations of [3, 4].

Let us stress that the BRST technique, originally conceived as a tool for quantization in the presence of a gauge symmetry [16], has proved over the years remarkably powerful also for formulating classical field theories. This type of application, initially proposed by Siegel [18], led promptly to the free String Field Theory constructions of [19], and shortly thereafter to the extension of the BRST analysis of [20] in the presence of open-string interactions [21, 22, 23]. More recently, this technique was widely used in [15, 17, 24, 25] to define significant instances of higher-spin dynamics in flat space and in (A)dS backgrounds. As we shall see, it has a direct bearing on the search for extensions of the triplets of [14, 4] and for the formulation of higher-spin dynamics with an unconstrained gauge symmetry. These results should be also of some interest in view of the potential relevance of higher-spin gauge theories [26, 10] for the AdS/CFT correspondence [27] in the weak gauge-coupling limit, a subject that recently has received an increasing attention and has also motivated some authors to reconsider the key properties of low-tension strings [28]. A related observation is that the BRST charge of world-sheet reparametrizations embodies a massive dynamics of the Fronsdal type, some aspects of which are manifest in the constructions of [29, 30, 31]. However, in this paper we shall confine our attention to the case of massless higher spin fields, leaving a detailed BRST analysis of massive higher spin fields propagating in (conformally) flat backgrounds for a future study.

Fermi fields also suggest a triplet-like structure, and indeed some of the excitations present in fermionic strings are described precisely by the fermionic triplets of symmetric spinor-tensors proposed in [4]. These comprise a spin- $s$  field  $\psi$ , a spin- $(s-1)$  field  $\chi$  and a spin- $(s-2)$  field  $\lambda$ , and if the tensor indices are left implicit the corresponding equations

read

$$\begin{aligned}\partial\psi &= \partial\chi, \\ \partial\chi &= \partial\cdot\psi - \partial\lambda, \\ \partial\lambda &= \partial\cdot\chi.\end{aligned}\tag{1.6}$$

Differently from the bosonic triplets, these systems propagate *all* half-odd integer spin chains up to and including a given one. After recovering this structure in the NSR string, we shall be able to deduce from it local compensator equations for all higher-spin fermions, both in flat space and in an (A)dS background, although the triplets themselves, for a reason that will become clear in the BRST analysis presented in Section 5, do not allow direct Lagrangian (A)dS extensions. Since an off-shell formulation for higher-spin fermions is being constructed by other authors [32], we shall refrain from completing the relevant steps in this case. All in all, we can well conclude that the BRST formalism proves once more quite powerful in dealing with these constrained systems, and provides a straight path toward the construction of consistent field equations and Lagrangians.

The content of the present paper is as follows. In Section 2 we discuss how triplets of symmetric tensors emerge from the bosonic string in the low-tension limit and describe their generalizations to tensors with mixed symmetry. In Section 3 we present their extension to (A)dS backgrounds, proceeding in two ways, first by a direct computation and then adapting the BRST analysis to this case, since this clarifies the very reason behind the consistency of the construction. In Section 4 we then turn to local field equations, in flat space and in (A)dS backgrounds, for individual higher-spin bosons with the unconstrained gauge symmetry of [3, 4], both in a reduced Vasiliev-like form and in a complete off-shell form motivated by [17]. Finally, in Section 5 we describe the extension of these results to fermions, recovering the corresponding triplets from the NSR string, explaining why, rather surprisingly, they do not extend to (A)dS backgrounds and deriving from them local non-Lagrangian equations with the unconstrained gauge symmetry of [3].

## 2. The bosonic string triplets

In this Section we describe how the tensionless limit of the free bosonic string exhibits the triplets of symmetric tensors of [14, 4]. This does not correspond directly to the behavior of tensionless strings, where the limit is taken prior to first quantization, a subject pioneered in [33]. We also display their generalization to the case of tensors with mixed symmetry, thus completing the description of the open bosonic string spectrum in the tensionless limit. Actually, although we shall deal explicitly with the open bosonic string, the closed bosonic string will be also fully encompassed by our discussion of generalized triplets in subsection 2.4.

### 2.1. The open bosonic oscillators and the Virasoro algebra

In order to set up our notation, let us begin by recalling some standard properties of the open bosonic string oscillators, that in the “mostly positive” space-time signature we shall adopt throughout satisfy the commutation relations

$$[\alpha_k^\mu, \alpha_l^\nu] = k \delta_{k+l,0} \eta^{\mu\nu}. \quad (2.1)$$

The corresponding Virasoro generators

$$L_k = \frac{1}{2} \sum_{l=-\infty}^{+\infty} \alpha_{k-l}^\mu \alpha_{\mu l}, \quad (2.2)$$

where  $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$  and  $p_\mu = -i\partial_\mu$ , satisfy the Virasoro algebra

$$[L_k, L_l] = (k-l) L_{k+l} + \frac{\mathcal{D}}{12} m(m^2 - 1), \quad (2.3)$$

where  $\mathcal{D}$  denotes the total space-time dimension.

In this paper we are interested in the tensionless limit, where the full gauge symmetry of the massive string spectrum is recovered, and to this end it is convenient to define the reduced generators

$$\ell_0 = p^2, \quad \ell_m = p \cdot \alpha_m \quad (m \neq 0). \quad (2.4)$$

They are related by suitable rescalings to the naive tensionless limit of the Virasoro generators, and satisfy the simpler algebra

$$[\ell_k, \ell_l] = k \delta_{k+l,0} \ell_0, \quad (2.5)$$

where the central charge has disappeared.

## 2.2. The BRST charge and the tensionless limit

It is also convenient to introduce the ghost modes  $C_k$ , of ghost number  $g = 1$ , and the corresponding anti-ghost modes  $B_k$ , of ghost number  $g = -1$ , with the anti-commutation relations

$$\{C_k, B_l\} = \delta_{k+l, 0} . \quad (2.6)$$

Indeed the BRST operator, first constructed in [20], that in this case is

$$\mathcal{Q} = \sum_{-\infty}^{+\infty} \left[ C_{-k} L_k - \frac{1}{2}(k-l) : C_{-k} C_{-l} B_{k+l} : \right] - C_0 , \quad (2.7)$$

determines the free string field equation [20, 21, 22]

$$\mathcal{Q} |\Phi\rangle = 0 , \quad (2.8)$$

where for the open bosonic string  $|\Phi\rangle$  has ghost number  $g = -1/2$ , while the corresponding ghost vacuum satisfies the conditions

$$\begin{aligned} B_0 |0\rangle_{gh} &= 0 , \\ B_k |0\rangle_{gh} &= 0 \quad (k > 0) , \\ C_k |0\rangle_{gh} &= 0 \quad (k > 0) , \end{aligned} \quad (2.9)$$

and actually a similar form, with the proper BRST operator, applies to all types of strings [35]. The nilpotency of  $\mathcal{Q}$  in the critical dimension ( $\mathcal{D} = 26$ ) implies the existence of an infinite chain of nested gauge symmetries

$$\delta |\Phi\rangle = \mathcal{Q} |\Lambda\rangle , \quad \delta |\Lambda\rangle = \mathcal{Q} |\tilde{\Lambda}\rangle , \quad \dots \quad (2.10)$$

that are typical of systems of forms, and at the same time guarantees the consistency of eq. (2.8).

Rescaling the ghost variables according to

$$c_k = \sqrt{2 \alpha'} C_k , \quad b_k = \frac{1}{\sqrt{2 \alpha'}} B_k \quad (2.11)$$

for  $k \neq 0$ , and as

$$c_0 = \alpha' C_0, \quad b_0 = \frac{1}{\alpha'} B_0 \quad (2.12)$$

for  $k = 0$ , does not affect their anti-commutation relations, but allows a non-singular  $\alpha' \rightarrow \infty$  limit that defines the *identically* nilpotent BRST charge

$$Q = \sum_{-\infty}^{+\infty} \left[ c_{-k} \ell_k - \frac{k}{2} b_0 c_{-k} c_k \right]. \quad (2.13)$$

We have thus recalled two equivalent manifestations of the tensionless limit, in the constraint algebra and in the BRST charge. The latter choice will prove particularly convenient, and affords interesting generalizations we shall return to repeatedly in the following sections.

It is convenient to write  $Q$  concisely as

$$Q = c_0 \ell_0 - b_0 M + \tilde{Q}, \quad (2.14)$$

with

$$\tilde{Q} = \sum_{k \neq 0} c_{-k} \ell_k \quad \text{and} \quad M = \frac{1}{2} \sum_{-\infty}^{+\infty} k c_{-k} c_k. \quad (2.15)$$

In a similar fashion, the string field  $|\Phi\rangle$  and the gauge parameter  $|\Lambda\rangle$  can be decomposed as

$$|\Phi\rangle = |\varphi_1\rangle + c_0 |\varphi_2\rangle, \quad (2.16)$$

$$|\Lambda\rangle = |\Lambda_1\rangle + c_0 |\Lambda_2\rangle, \quad (2.17)$$

and as a result the field equations and the corresponding gauge transformations become

$$\begin{aligned} \ell_0 |\varphi_1\rangle - \tilde{Q} |\varphi_2\rangle &= 0, \\ \tilde{Q} |\varphi_1\rangle - M |\varphi_2\rangle &= 0, \end{aligned} \quad (2.18)$$

$$\begin{aligned} \delta |\varphi_1\rangle &= \tilde{Q} |\Lambda_1\rangle - M |\Lambda_2\rangle, \\ \delta |\varphi_2\rangle &= \ell_0 |\Lambda_1\rangle - \tilde{Q} |\Lambda_2\rangle. \end{aligned} \quad (2.19)$$

It should be appreciated that these field equations are consistent and gauge invariant *in any space-time dimension*. This is to be contrasted with the ordinary equations for the

tensile string where, as is well known, the critical space-time dimension  $\mathcal{D} = 26$  plays a crucial role in allowing a consistent mass generation.

### 2.3. The case of symmetric tensors

Let us now confine our attention to totally symmetric tensors, thus working only with the  $(\alpha_{-1}, \alpha_1)$  oscillator pair and effectively reducing the constraints to the  $(\ell_{-1}, \ell_0, \ell_1)$  triplet. As a result, the string field  $|\Phi\rangle$  and the gauge parameter  $|\Lambda\rangle$  involve only the three ghost modes  $(c_{-1}, c_0, c_1)$  and the corresponding anti-ghost modes  $(b_{-1}, b_0, b_1)$ , while the ghost vacuum satisfies the conditions

$$c_1|0\rangle_{gh} = 0, \quad b_1|0\rangle_{gh} = 0, \quad b_0|0\rangle_{gh} = 0. \quad (2.20)$$

The limiting form of  $Q$  then implies that the field equations describe independent *triplets*  $(\varphi, C, D)$  of symmetric tensors of ranks  $(s, s-1, s-2)$ , defined via

$$\begin{aligned} |\varphi_1\rangle &= \frac{1}{s!} \varphi_{\mu_1 \dots \mu_s}(x) \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_s} |0\rangle \\ &\quad + \frac{1}{(s-2)!} D_{\mu_1 \dots \mu_{s-2}}(x) \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_{s-2}} c_{-1} b_{-1} |0\rangle, \\ |\varphi_2\rangle &= \frac{-i}{(s-1)!} C_{\mu_1 \dots \mu_{s-1}}(x) \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_{s-1}} b_{-1} |0\rangle, \end{aligned} \quad (2.21)$$

while the corresponding gauge transformation parameters  $|\Lambda\rangle$ ,

$$|\Lambda\rangle = \frac{i}{(s-1)!} \Lambda_{\mu_1 \mu_2 \dots \mu_{s-1}}(x) \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_{s-1}} b_{-1} |0\rangle, \quad (2.22)$$

describe symmetric tensors of rank  $(s-1)$ .

In dealing with these systems of symmetric tensors, it is convenient to resort to the compact notation of [3, 4], thus omitting all indices carried by the totally symmetric triplet fields, by the Minkowski metric tensor and by space-time derivatives. One can then proceed rather simply, but for a few seemingly unfamiliar combinatoric rules [3, 4], that we collect for later use,

$$(\partial^p \varphi)' = \square \partial^{p-2} \varphi + 2 \partial^{p-1} \partial \cdot \varphi + \partial^p \varphi', \quad (2.23)$$

$$\partial^p \partial^q = \binom{p+q}{p} \partial^{p+q}, \quad (2.24)$$

$$\partial \cdot (\partial^p \varphi) = \square \partial^{p-1} \varphi + \partial^p \partial \cdot \varphi, \quad (2.25)$$

$$\partial \cdot \eta^k = \partial \eta^{k-1}, \quad (2.26)$$

$$(\eta^k T_{(s)})' = k [\mathcal{D} + 2(s+k-1)] \eta^{k-1} T_{(s)} + \eta^k T'_{(s)}, \quad (2.27)$$

where  $\mathcal{D}$  denotes the total space-time dimension and  $T_{(s)}$  is a generic symmetric rank- $s$  tensor.

Expanding (2.18) and (2.19) then leads to the *triplet* equations of [4]

$$\begin{aligned} \square \varphi &= \partial C, \\ \partial \cdot \varphi - \partial D &= C, \\ \square D &= \partial \cdot C, \end{aligned} \quad (2.28)$$

and to the corresponding gauge transformations

$$\begin{aligned} \delta \varphi &= \partial \Lambda, \\ \delta C &= \square \Lambda, \\ \delta D &= \partial \cdot \Lambda. \end{aligned} \quad (2.29)$$

Let us stress that here  $\Lambda$  is an *unconstrained* parameter, to be contrasted with the *traceless* gauge parameter of the Fronsdal formulation of higher-spin gauge fields [1]. Interestingly, this type of structure was first exhibited long ago in the tensionless limit of the open bosonic string, for the first two massive levels, by A. Bengtsson [14], in an equivalent form without the field  $C$ . As discussed in [14, 4], these equations propagate modes of spin  $s$ ,  $s-2$ , ..., down to zero or one according to whether  $s$  is even or odd. This makes up a total of  $\binom{\mathcal{D}+s-3}{s}$  degrees of freedom if  $\mathcal{D} > 4$ , or simply  $(s+1)$  degrees of freedom if  $\mathcal{D} = 4$ . Nevertheless, as we shall see, these systems have a lot to teach us about irreducible higher-spin propagation. They were also considered in [15] as a particularly simple application of the BRST technique to describe massive fields via dimensional reduction.

It is interesting to note that the combinatorial identity

$$\binom{\mathcal{D}+s-2}{s} = \sum_{k=0}^s \binom{\mathcal{D}+k-3}{k} \quad (2.30)$$

suggests a mechanism of mass generation whereby a triplet gains mass swallowing a chain of other triplets of lower maximum spins. However, while such a nice and simple mechanism

indeed applies to the massive Kaluza-Klein modes originating from a  $\mathcal{D}+1 \rightarrow \mathcal{D}$  reduction, it cannot be held directly responsible for the mass generation in String Theory, where the mechanism takes place also within a single triplet, so that in fact the triplet structure is well hidden in tensile string spectra.

These field equations follow from the Lagrangian

$$\mathcal{L} = \langle \Phi | Q | \Phi \rangle , \quad (2.31)$$

that in component notation reads

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (\partial_\mu \varphi)^2 + s \partial \cdot \varphi C + s(s-1) \partial \cdot C D \\ & + \frac{s(s-1)}{2} (\partial_\mu D)^2 - \frac{s}{2} C^2 , \end{aligned} \quad (2.32)$$

where the  $D$  field, whose modes disappear on the mass shell, has a peculiar negative kinetic term. Alternatively, one can eliminate the auxiliary field  $C$ , thus obtaining an equivalent formulation in terms of *pairs*  $(\varphi, D)$  of symmetric tensors, more in the spirit of [14]. In terms of the Fronsdal kinetic operators

$$\mathcal{F} = \square \varphi - \partial \partial \cdot \varphi + \partial^2 \varphi' , \quad (2.33)$$

that satisfy the Bianchi identities

$$\partial \cdot \mathcal{F} - \frac{1}{2} \partial \mathcal{F}' = -\frac{3}{2} \partial^3 \varphi'' , \quad (2.34)$$

the field equations then become

$$\begin{aligned} \mathcal{F} = & \partial^2 (\varphi' - 2D) , \\ \square D = & \frac{1}{2} \partial \cdot \partial \cdot \varphi - \frac{1}{2} \partial \partial \cdot D , \end{aligned} \quad (2.35)$$

and follow from the Lagrangians

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (\partial_\mu \varphi)^2 + \frac{s}{2} (\partial \cdot \varphi)^2 + s(s-1) \partial \cdot \partial \cdot \varphi D \\ & + s(s-1) (\partial_\mu D)^2 + \frac{s(s-1)(s-2)}{2} (\partial \cdot D)^2 . \end{aligned} \quad (2.36)$$

## 2.4. Generalized bosonic triplets of mixed symmetry

It is actually not difficult to account for more general tensors of mixed symmetry resulting from the interplay of  $r$  types of string oscillators.<sup>1</sup> The totally symmetric rank- $s$  field  $\varphi$  is then replaced by more general gauge fields  $\varphi$  with  $r$  sets of  $n_1, \dots, n_r$  totally symmetric indices, such that  $\sum_{k=1}^r n_k = s$ , and the resulting system will describe a total of

$$\prod_{k=1}^r \binom{\mathcal{D} + n_k - 3}{n_k} \quad (2.37)$$

degrees of freedom. The natural guess would then be that the single auxiliary  $C$  field be replaced by  $r$  auxiliary fields  $C^i$  ( $i = 1, \dots, r$ ), and finally that the single  $D$  field be replaced by  $r^2$  additional fields  $D_i^j$  ( $i, j = 1, \dots, r$ ).

The resulting gauge transformations should thus be

$$\begin{aligned} \delta \varphi &= \sum_{i=1}^r \partial^i \Lambda^i , \\ \delta C^i &= \square \Lambda^i , \\ \delta D^{ij} &= \partial^i \cdot \Lambda^j , \end{aligned} \quad (2.38)$$

where  $\partial^i$  denotes a derivative with respect to an index of the  $i$ -th set, so that the corresponding field equations

$$\begin{aligned} \square \varphi &= \sum_{i=1}^r \partial^i C^i , \\ \partial^i \cdot \varphi - \sum_{j=1}^r \partial^j D^{ij} &= C^i , \\ \square D^{ij} &= \partial^i \cdot C^j , \end{aligned} \quad (2.39)$$

would be the natural generalization of eq. (2.28).

The proper description of this system, however, requires a constraint,

$$\partial^k \cdot D^{ij} = \partial^i \cdot D^{kj} , \quad (2.40)$$

whose emergence can be anticipated since the two apparently distinct expressions transform identically under the gauge transformations (2.38). This constraint, instrumental in

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<sup>1</sup>An early BRST treatment of these systems may be found in [34]. We are grateful to G. Bonelli for calling this reference to our attention.

attaining the elimination of all unwanted field components, has an important consequence: in general (2.39) and (2.40) *are not Lagrangian equations*, since one is missing at least the Lagrange multipliers needed to enforce (2.40). This is the first instance of a phenomenon that we shall meet again in the following, since indeed only weaker conditions follow from the integrability of the second of (2.39).

For instance, for a field  $\varphi_{\mu\nu,\rho\sigma}$ , symmetric only under the interchange of the indices within the two sets, eqs. (2.39) become

$$\begin{aligned}
 \square \varphi_{\mu\nu,\rho\sigma} &= \partial_\mu C_{\nu,\rho\sigma}^1 + \partial_\nu C_{\mu,\rho\sigma}^1 + \partial_\rho C_{\mu\nu,\sigma}^2 + \partial_\sigma C_{\mu\nu,\rho}^2 , \\
 C_{\nu,\rho\sigma}^1 &= \partial^\mu \varphi_{\mu\nu,\rho\sigma} - \partial_\nu D_{\rho\sigma}^{11} - \partial_\rho D_{\nu,\sigma}^{12} - \partial_\sigma D_{\nu,\rho}^{12} , \\
 C_{\mu\nu,\sigma}^2 &= \partial^\rho \varphi_{\mu\nu,\rho\sigma} - \partial_\sigma D_{\mu\nu}^{22} - \partial_\mu D_{\nu,\sigma}^{21} - \partial_\nu D_{\mu,\sigma}^{21} , \\
 \square D_{\rho\sigma}^{11} &= \partial^\nu C_{\nu,\rho\sigma}^1 , \\
 \square D_{\nu,\sigma}^{12} &= \partial^\mu C_{\mu\nu,\sigma}^2 , \\
 \square D_{\nu,\sigma}^{21} &= \partial^\rho C_{\nu,\rho\sigma}^1 , \\
 \square D_{\mu\nu}^{22} &= \partial^\sigma C_{\mu\nu,\sigma}^2 , \tag{2.41}
 \end{aligned}$$

where each ',' separates two different groups of totally symmetric space-time indices. The second and third of these then lead to the integrability constraint

$$\partial_\nu \partial^\rho D_{\rho\sigma}^{11} + \partial^\rho \partial_\sigma D_{\nu,\rho}^{12} = \partial^\mu \partial_\sigma D_{\mu\nu}^{22} + \partial^\mu \partial_\nu D_{\mu,\sigma}^{21} , \tag{2.42}$$

weaker than eq. (2.40), that in this case would lead to the two conditions

$$\begin{aligned}
 \partial^\mu D_{\mu,\sigma}^{21} &= \partial^\rho D_{\rho\sigma}^{11} , \\
 \partial^\mu D_{\mu\nu}^{22} &= \partial^\rho D_{\nu,\rho}^{12} , \tag{2.43}
 \end{aligned}$$

that clearly imply (2.42).

The BRST technique leads nicely to a solution of the problem and to an off-shell formulation for these generalized triplets, albeit in terms of a wider set of fields. To this end, one has to resort to a family of  $\alpha_{-i}^\mu$  oscillators ( $i = 1, \dots, r$ ), that are needed to build tensors of this general type, and these bring about corresponding (anti)ghosts  $(b_{\pm i})c_{\pm i}$ . The result

is a wider, but still *finite*, set of fields that generalize the naive  $\varphi$ ,  $C_i$  and  $D_{ij}$ , collectively written as

$$|\Phi\rangle = \frac{c_{-i_1} \dots c_{-i_l} b_{-j_1} \dots b_{-j_l}}{(l!)^2} |D_{i_1 \dots i_l}^{j_1 \dots j_l}\rangle + \frac{c_0 c_{-i_1} \dots c_{-i_{l-1}} b_{-j_1} \dots b_{-j_l}}{(l-1)! l!} |C_{i_1 \dots i_{l-1}}^{j_1 \dots j_l}\rangle , \quad (2.44)$$

where each  $C$  and  $D$  “ket” depends on the bosonic oscillators  $\alpha_{-i}^\mu$ , and where the original  $\varphi$  field is described by the  $|D\rangle$  field carrying no (anti)ghost indices, while the original  $C_i$  are described by the  $|C\rangle$  fields with the lowest number of indices, that in their case is indeed a single anti-ghost index due to the  $c_0$  factor. The individual terms contain variable numbers of  $\alpha_{-i}$  oscillators, as required by the structure of the field  $\varphi$  one is trying to describe, so that each index  $i_p$  (or  $j_p$ ) carried by the  $C$  or  $D$  fields reduces by one unit the corresponding number of  $\alpha_{-i_p}$  (or  $\alpha_{-j_p}$ ) oscillators.

The corresponding gauge parameters can be collectively written

$$\begin{aligned} |\Lambda^{(1)}\rangle &= \frac{c_{-i_1} \dots c_{-i_l} b_{-j_1} \dots b_{-j_{l+1}}}{l!(l+1)!} |\Lambda_{i_1 \dots i_l}^{1(1) j_1 \dots j_{l+1}}\rangle \\ &+ \frac{c_0 c_{-i_1} \dots c_{-i_{l-1}} b_{-j_1} \dots b_{-j_{l+1}}}{(l-1)!(l+1)!} |\Lambda_{i_1 \dots i_{l-1}}^{2(1) j_1 \dots j_{l+1}}\rangle , \end{aligned} \quad (2.45)$$

to distinguish them from the “gauge-for-gauge” parameters  $|\Lambda^{(p)}\rangle$  (with  $p > 1$ ), that are now present. The resulting field equations

$$\begin{aligned} \ell_0 |D_{i_1 \dots i_l}^{j_1 \dots j_l}\rangle + (-1)^l \ell_{i_l} |C_{i_1 \dots i_{l-1}}^{j_1 \dots j_l}\rangle - (-1)^l \ell_{-j} |C_{i_1 \dots i_l}^{j, j_1 \dots j_l}\rangle &= 0 , \\ \ell_{i_l} |D_{i_1 \dots i_{l-1}}^{j_1 \dots j_{l-1}}\rangle - \ell_{-j} |D_{i_1 \dots i_l}^{j, j_1 \dots j_{l-1}}\rangle + (-1)^l |C_{i_1 \dots i_{l-1}}^{i_l j_1 \dots j_{l-1}}\rangle &= 0 , \end{aligned} \quad (2.46)$$

are thus invariant under the gauge transformations

$$\begin{aligned} \delta |D_{i_1 \dots i_l}^{j_1 \dots j_l}\rangle &= -(-1)^l \ell_{i_l} |\Lambda_{i_1 \dots i_{l-1}}^{1(1) j_1 \dots j_l}\rangle + (-1)^l \ell_{-j} |\Lambda_{i_1 \dots i_l}^{1(1) j j_1 \dots j_l}\rangle - |\Lambda_{i_1 \dots i_{l-1}}^{2(1) i_l j_1 \dots j_l}\rangle , \\ \delta |C_{i_1 \dots i_{l-1}}^{j_1 \dots j_l}\rangle &= \ell_0 |\Lambda_{i_1 \dots i_{l-1}}^{1(1) j_1 \dots j_l}\rangle - (-1)^l \ell_{i_{l-1}} |\Lambda_{i_1 \dots i_{l-2}}^{2(1) j_1 \dots j_l}\rangle + (-1)^l \ell_{-j} |\Lambda_{i_1 \dots i_{l-1}}^{(1) j j_1 \dots j_l}\rangle , \end{aligned} \quad (2.47)$$

that, in their turn, are invariant under the chain of “gauge-for-gauge” transformations

$$\begin{aligned} \delta |\Lambda_{i_1 \dots i_l}^{1(k) j_1 \dots j_{l+k}}\rangle &= -(-1)^l \ell_{i_l} |\Lambda_{i_1 \dots i_{l-1}}^{1(k+1) j_1 \dots j_{l+k}}\rangle \\ &+ (-1)^l \ell_{-j} |\Lambda_{i_1 \dots i_l}^{1(k+1) j j_1 \dots j_{l+k}}\rangle - |\Lambda_{i_1 \dots i_{l-1}}^{2(k+1) i_l j_1 \dots j_{l+k}}\rangle , \\ \delta |\Lambda_{i_1 \dots i_{l-1}}^{2(k) j_1 \dots j_{l+k}}\rangle &= \ell_0 |\Lambda_{i_1 \dots i_{l-1}}^{1(k+1) j_1 \dots j_{l+k}}\rangle \\ &- (-1)^l \ell_{i_{l-1}} |\Lambda_{i_1 \dots i_{l-2}}^{2(k+1) j_1 \dots j_{l+k}}\rangle + (-1)^l \ell_{-j} |\Lambda_{i_1 \dots i_{l-1}}^{2(k+1) j j_1 \dots j_{l+k}}\rangle , \end{aligned} \quad (2.48)$$

and so on.

Partial gauge fixing of this system reduces it to (2.39), but the BRST technique provides directly an off-shell description, and leads to the gauge-invariant Lagrangians

$$\begin{aligned} \mathcal{L} = & - \sum_l \left[ \frac{(-1)^l}{(l!)^2} \langle D_{j_1 \dots j_l}^{i_1 \dots i_l} | \ell_0 | D_{i_1 \dots i_l}^{j_1 \dots j_l} \rangle + \frac{2}{(l-1)! l!} \langle C_{i_1 \dots i_{l-1}}^{j_1 \dots j_l} | \ell_{-i_l} | D_{j_1 \dots j_l}^{i_1 \dots i_l} \rangle \right. \\ & \left. + \frac{2 (-1)^l}{(l!)^2} \langle C_{i_1 \dots i_l}^{j_1 \dots j_{l+1}} | \ell_{j_{l+1}} | D_{j_1 \dots j_l}^{i_1 \dots i_l} \rangle - \frac{(-1)^l}{((l-1)!)^2} \langle C_{j_1 \dots j_{l-1}}^{i_1 \dots i_{l-1} j_l} | C_{i_1 \dots i_{l-1}}^{j_1 \dots j_l} \rangle \right]. \end{aligned} \quad (2.49)$$

For the tensor  $\varphi_{\mu\nu,\rho\sigma}$  considered above, the BRST analysis introduces the previous fields  $C_{\nu,\rho\sigma}^1$ ,  $C_{\mu\nu,\rho}^2$ ,  $D_{\rho\sigma}^{11}$ ,  $D_{\nu,\sigma}^{12}$ ,  $D_{\mu,\rho}^{21}$  and  $D_{\mu\nu}^{22}$ , together with the additional ones,  $C_\rho^{112}$ ,  $C_\mu^{122}$  and  $D^{1212}$ , so that the resulting equations include (2.39), that are not modified, together with additional ones,

$$\begin{aligned} \partial^\rho D_{\rho\sigma}^{11} - \partial^\nu D_{\nu\sigma}^{21} &= C_\sigma^{112} + \partial_\sigma D^{1212}, \\ \partial^\rho D_{\nu\rho}^{12} - \partial^\mu D_{\mu\nu}^{22} &= C_\nu^{122} - \partial^\nu D^{1212}, \\ \square D^{1212} &= \partial^\rho C_\rho^{112} - \partial^\mu C_\mu^{122}, \end{aligned} \quad (2.50)$$

for the new fields  $C_\rho^{112}$ ,  $C_\mu^{122}$  and  $D^{1212}$ . Eqs. (2.39) and (2.50) are now invariant under the modified gauge transformations

$$\begin{aligned} \delta \varphi_{\mu\nu,\rho\sigma} &= \partial_\mu \Lambda_{\nu,\rho\sigma}^1 + \partial_\nu \Lambda_{\mu,\rho\sigma}^1 + \partial_\rho \Lambda_{\mu\nu,\sigma}^2 + \partial_\sigma \Lambda_{\mu\nu,\rho}^2, \\ \delta C_{\mu,\rho\sigma}^1 &= \square \Lambda_{\mu,\rho\sigma}^1, \\ \delta C_{\mu\nu,\rho}^2 &= \square \Lambda_{\mu\nu,\sigma}^2, \\ \delta D_{\rho\sigma}^{11} &= \partial^\mu \Lambda_{\mu,\rho\sigma}^1 + \partial_\rho \Lambda_\sigma^{112} + \partial_\sigma \Lambda_\rho^{112}, \\ \delta D_{\mu,\rho}^{12} &= \partial^\nu \Lambda_{\mu\nu,\rho}^2 - \partial_\mu \Lambda_\rho^{112}, \\ \delta D_{\mu,\rho}^{21} &= \partial^\sigma \Lambda_{\mu,\rho\sigma}^1 + \partial_\rho \Lambda_\mu^{122}, \\ \delta D_{\mu\nu}^{22} &= \partial^\rho \Lambda_{\mu\nu,\rho}^2 - \partial_\mu \Lambda_\nu^{122} - \partial_\nu \Lambda_\mu^{122}. \end{aligned} \quad (2.51)$$

A partial gauge fixing of eqs. (2.39) and (2.50), making use of the new gauge parameters, reduces the fields to the naive set, but the constraints (2.40) now indeed emerge as additional field equations introduced by the BRST procedure.

This more general set of fields exhausts the spectrum of the open bosonic string in the tensionless limit. Actually, the generalized triplets (2.38) exhaust all cases presented by the closed bosonic string as well, since for the closed string the relevant states involve at least two sets of oscillators associated with left and right world-sheet modes. Hence, the spectra of all bosonic models in the tensionless limit are built out of an infinite collection of these (generalized) triplets.

### 3. (A)dS extension of bosonic string triplets

In this Section we describe how to construct the (A)dS extension of the massless triplets that have emerged from the bosonic string in the tensionless limit, but for brevity we confine our attention to symmetric triplets. It is well known that higher-spin gauge fields propagate consistently and independently of one another in conformally flat space times, thus bypassing the well-known Aragone-Deser inconsistencies [11] that would be introduced by a background Weyl tensor. The bosonic triplets also allow this extension rather simply, but it is instructive to see how the story develops.

#### 3.1. Direct construction

One can construct directly the (A)dS extensions of the bosonic triplets, starting from the flat-space equations (2.28) and (2.29). While the gauge transformations of  $\varphi$  and  $D$  are naturally turned into their curved-space counterparts

$$\begin{aligned} \delta\varphi &= \nabla\Lambda , \\ \delta D &= \nabla \cdot \Lambda , \end{aligned} \tag{3.1}$$

the key observation is to deduce the deformed transformation of  $C$  from the condition that the constraint relating it to  $\varphi$  and  $D$ ,

$$C = \nabla \cdot \varphi - \nabla D , \tag{3.2}$$

be retained.

The result,

$$\delta C = \square\Lambda + \frac{(s-1)(3-s-\mathcal{D})}{L^2}\Lambda + \frac{2}{L^2}g\Lambda' , \tag{3.3}$$

where “primes” as usual denote traces,  $\mathcal{D}$  is the space-time dimension,  $L^2$  determines the (A)dS cosmological constant,  $g$  denotes the background metric tensor and  $s$  denotes the spin of  $\varphi$ , then fixes unambiguously the form of the other equations. All this rests on the only new datum of the deformed problem, the commutator of two covariant derivatives on a vector,

$$[\nabla_\mu, \nabla_\nu] V_\rho = \frac{1}{L^2} (g_{\nu\rho} V_\mu - g_{\mu\rho} V_\nu) . \quad (3.4)$$

This result actually applies to AdS, while the corresponding one for a dS background can be formally recovered continuing  $L^2$  to negative values.

The gauge transformations (3.1) and (3.3) determine completely the resulting (A)dS equations, that can be presented in the rather compact form

$$\begin{aligned} \square \varphi &= \nabla C + \frac{1}{L^2} \left\{ 8gD - 2g\varphi' + [(2-s)(3-\mathcal{D}-s) - s]\varphi \right\} , \\ C &= \nabla \cdot \varphi - \nabla D , \\ \square D &= \nabla \cdot C + \frac{1}{L^2} \left\{ [s(\mathcal{D}+s-2) + 6]D - 4\varphi' - 2gD' \right\} . \end{aligned} \quad (3.5)$$

As in the previous section, one can also eliminate  $C$ . To this end, it is convenient to define the *AdS* Fronsdal operator

$$\mathcal{F} = \square \varphi - \nabla \nabla \cdot \varphi + \frac{1}{2} \{\nabla, \nabla\} \varphi' , \quad (3.6)$$

and the first equation then becomes

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} \{\nabla, \nabla\} (\varphi' - 2D) + \frac{1}{L^2} \left\{ 8gD - 2g\varphi' \right. \\ &\quad \left. + [(2-s)(3-\mathcal{D}-s) - s]\varphi \right\} . \end{aligned} \quad (3.7)$$

In a similar fashion, after eliminating the auxiliary field  $C$  the (A)dS equation for  $D$  becomes

$$\begin{aligned} \square D + \frac{1}{2} \nabla \nabla \cdot D - \frac{1}{2} \nabla \cdot \nabla \cdot \varphi &= - \frac{(s-2)(4-\mathcal{D}-s)}{2L^2} D - \frac{1}{L^2} g D' \\ &\quad + \frac{1}{2L^2} \left\{ [s(\mathcal{D}+s-2) + 6]D - 4\varphi' - 2gD' \right\} . \end{aligned} \quad (3.8)$$

It is also convenient to elaborate further on these expressions, defining the modified Fronsdal operator

$$\mathcal{F}_L = \mathcal{F} - \frac{1}{L^2} \left\{ [(3-\mathcal{D}-s)(2-s) - s]\varphi + 2g\varphi' \right\} , \quad (3.9)$$

since in terms of  $\mathcal{F}_L$  the deformed Bianchi identity (2.34) retains a rather simple form,

$$\nabla \cdot \mathcal{F}_L - \frac{1}{2} \nabla \mathcal{F}'_L = -\frac{3}{2} \nabla^3 \varphi'' + \frac{2}{L^2} g \nabla \varphi''. \quad (3.10)$$

In Section 4 we shall see how this Bianchi identity determines *local* non-Lagrangian higher-spin equations in (A)dS with the same unconstrained gauge symmetry present in the non-local geometric construction of [3, 4].

### 3.2. Consistency and the AdS BRST charge

We have thus seen how the triplets emerging from the bosonic string in the tensionless limit extend rather simply to the case of (A)dS backgrounds, although the tensile string spectrum does not display such a simple behavior for well-known reasons related to the central extension of the Virasoro algebra. It is very instructive to retrace these steps in the BRST formulation, since the resulting analysis clarifies the reasons behind the very consistency of the construction. Indeed, as we shall see in later sections, the fermionic triplets proposed in [4] can be derived from the tensionless limit of the fermionic string, but do not allow a similar Lagrangian (A)dS deformation for reasons that the BRST analysis will explain rather neatly.

The starting point for this discussion is the (A)dS form of the commutator of two covariant derivatives on a vector of eq. (3.4). In trying to adapt the BRST construction to this case, let us begin by introducing the tangent-space valued oscillators  $(\alpha_{-1}^a, \alpha_1^a)$ , that satisfy

$$[\alpha_1^a, \alpha_{-1}^b] = \eta^{ab}, \quad (3.11)$$

or the corresponding oscillators  $(\alpha_{-1}^\mu, \alpha_1^\mu)$ , obtained contracting them with the vielbein  $e_\mu^a$ , that satisfy

$$[\alpha_1^\mu, \alpha_{-1}^\nu] = g^{\mu\nu}, \quad (3.12)$$

where  $g$  denotes the (A)dS metric.

The ordinary partial derivative must now be replaced by an operator that, acting on the totally symmetric Fock-space tensors built from the single oscillator  $\alpha_{-1}$ , produces the proper covariant derivative. This operator, denoted in the following again by  $p_\mu$ , can be defined as

$$p_\mu = -i (\partial_\mu - \Gamma^\rho_{\mu\nu} \alpha_{-1}^\nu \alpha_{1\rho}), \quad (3.13)$$

or equivalently as

$$p_\mu = -i (\partial_\mu + \omega_\mu^{ab} \alpha_{-1a} \alpha_{1b}) , \quad (3.14)$$

where  $\Gamma$  and  $\omega$  denote the Christoffel and spin connections. It is then simple to verify that

$$[p_\mu, p_\nu] = \frac{1}{L^2} (\alpha_{-1\mu} \alpha_{1\nu} - \alpha_{-1\nu} \alpha_{1\mu}) , \quad (3.15)$$

since for an (A)dS space the Riemann tensor is simply

$$R_{\mu\nu\rho\sigma} = \frac{1}{L^2} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) . \quad (3.16)$$

In a similar fashion, one can see that

$$\ell_0 = g^{\mu\nu} (p_\mu p_\nu + i \Gamma_{\mu\nu}^\lambda p_\lambda) = p^a p_a - i \omega_a^{ab} p_b \quad (3.17)$$

acts on Fock-space tensors as the proper D'Alembertian operator.

In order to determine the (A)dS extension of the BRST charge (2.13), let us insist on retaining the two constraints associated to<sup>2</sup>

$$\ell_{\pm 1} = \alpha_{\pm 1} \cdot p , \quad (3.18)$$

now built with the covariant derivative operator (3.13) or, equivalently, (3.14). However, the commutator of  $\ell_1$  and  $\ell_{-1}$  does not generate  $\ell_0$  as in flat space. Rather,

$$[\ell_1, \ell_{-1}] = \tilde{\ell}_0 , \quad (3.19)$$

where the modified D'Alembertian is

$$\tilde{\ell}_0 = \ell_0 - \frac{1}{L^2} \left( -\mathcal{D} + \frac{\mathcal{D}^2}{4} + 4 M^\dagger M - N^2 + 2N \right) , \quad (3.20)$$

with  $\mathcal{D}$ , as in previous sections, the total space-time dimension. Here

$$N = \alpha_{-1} \cdot \alpha_1 + \frac{\mathcal{D}}{2} \quad (3.21)$$

is like the contribution of  $\alpha_{\pm 1}$  to the squared mass in the tensile  $L_0$  generator, and thus counts the number of indices of the Fock-space fields, up to the space-time dimension  $\mathcal{D}$ , while

$$M = \frac{1}{2} \alpha_1 \cdot \alpha_1 \quad (3.22)$$

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<sup>2</sup>The operators  $\ell_{\pm 1}$  are hermitian conjugates of one another with respect to the AdS integration measure.

is like the contribution of  $\alpha_1$  to the tensile  $L_2$  generator, and thus takes traces of the Fock-space fields.

The emergence of these new operators enlarges the algebra, that now includes the additional commutators

$$\begin{aligned} [M^\dagger, \ell_1] &= -\ell_{-1}, \\ [\tilde{\ell}_0, \ell_1] &= \frac{2}{L^2} \ell_1 - \frac{4}{L^2} N \ell_1 + \frac{8}{L^2} \ell_{-1} M, \\ [N, \ell_1] &= -\ell_1, \end{aligned} \tag{3.23}$$

and their hermitian conjugates, together with

$$\begin{aligned} [N, M] &= -2 M, \\ [M^\dagger, N] &= -2 M^\dagger, \\ [M^\dagger, M] &= -N, \end{aligned} \tag{3.24}$$

that define an  $SO(1, 2)$  subalgebra.

Notice that (3.23) and (3.24) is actually a *non-linear algebra*, and therefore the associated BRST charge should be naively constructed with the recipe of [36]. As in [24], however, this would introduce a larger set of ghosts and corresponding fields, going beyond the triplet structure. Thus, in the spirit of the flat limit for the triplet, *let us retain only the  $(\ell_{\pm 1}, \ell_0)$  constraints, treating (3.23) as an ordinary algebra where  $M$ ,  $M^\dagger$  and  $N$  play the role of “structure constants”*. Remarkably, this is possible and guarantees the Lagrangian nature of eqs. (3.5) for the deformed triplets, since the additional operators act “diagonally” on the triplet fields, their only effect being to mix them and to introduce in the resulting equations some coefficients that depend explicitly on the spin  $s$  and on the space-time dimension  $\mathcal{D}$ .

With this proviso, one can write the *identically nilpotent* BRST charge

$$\begin{aligned} Q &= c_0 \left( \tilde{\ell}_0 - \frac{4}{L^2} N + \frac{6}{L^2} \right) + c_1 \ell_{-1} + c_{-1} \ell_1 - c_{-1} c_1 b_0 \\ &\quad - \frac{6}{L^2} c_0 c_{-1} b_1 - \frac{6}{L^2} c_0 b_{-1} c_1 + \frac{4}{L^2} c_0 c_{-1} b_1 N + \frac{4}{L^2} c_0 b_{-1} c_1 N \\ &\quad - \frac{8}{L^2} c_0 c_{-1} b_{-1} M + \frac{8}{L^2} c_0 c_1 b_1 M^\dagger + \frac{12}{L^2} c_0 c_{-1} b_{-1} c_1 b_1. \end{aligned} \tag{3.25}$$

The nilpotency of  $Q$  ensures the consistency of the construction, and as usual determines a BRST invariant Lagrangian of the form (2.31), and thus a Lagrangian set of equations as in (2.8). In component notation

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (\nabla_\mu \varphi)^2 + s \nabla \cdot \varphi C + s(s-1) \nabla \cdot C D + \frac{s(s-1)}{2} (\nabla_\mu D)^2 - \frac{s}{2} C^2 \\ & + \frac{s(s-1)}{2L^2} (\varphi')^2 - \frac{s(s-1)(s-2)(s-3)}{2L^2} (D')^2 - \frac{4s(s-1)}{L^2} D \varphi' \\ & - \frac{1}{2L^2} [(s-2)(\mathcal{D} + s - 3) - s] \varphi^2 + \frac{s(s-1)}{2L^2} [s(\mathcal{D} + s - 2) + 6] D^2, \end{aligned} \quad (3.26)$$

whose field equations are indeed those in (3.5).

## 4. Compensator form of the bosonic higher-spin equations

In this Section we show how one can obtain local non-Lagrangian descriptions of higher-spin bosons that exhibit the unconstrained gauge symmetry present in the non-local geometric equations of [3, 4] and reduce to the Fronsdal form after a partial gauge fixing. The triplets are actually very useful in this respect, since they suggest directly the form of the resulting equations. One can also arrive at more complicated fully gauge invariant Lagrangian formulations for higher-spin bosons, that are nicely determined by an extension of the BRST method discussed in the previous sections, obtained enlarging the constraint algebra as in [17]. Whereas the resulting equations were there connected to the Fronsdal formulation, here we shall see that a suitable partial gauge fixing and the corresponding judicious elimination of a number of auxiliary fields recovers the unconstrained gauge symmetry of [3, 4], and thus the non-Lagrangian equations presented in the next subsection.

### 4.1. Non-Lagrangian formulation in flat space

The case of a single propagating spin- $s$  field can be recovered from the results of the previous section restricting the attention to field configurations such that all lower-spin excitations are pure gauge. To this end, it suffices to demand that

$$\varphi' - 2D = \partial \alpha, \quad (4.1)$$

where  $\alpha$  is a spin- $(s-3)$  field that will play the role of the single compensator needed in this formulation. This choice, motivated by the fact that  $\varphi' - 2D$  transforms as a canonical

spin- $(s - 2)$  field, turns the first of eqs. (2.35) into

$$\mathcal{F} = 3\partial^3\alpha , \quad (4.2)$$

while the second eq. (2.35) takes an apparently more complicated form, and becomes

$$\square\varphi' + \frac{1}{2}\partial\partial\cdot\varphi' - \partial\cdot\partial\cdot\varphi = \frac{3}{2}\square\partial\alpha + \partial^2\partial\cdot\alpha . \quad (4.3)$$

In terms of the Fronsdal operator defined in eq. (2.33), however, this simplifies considerably, since (2.33) implies that

$$\mathcal{F}' = 2\square\varphi' - 2\partial\cdot\partial\cdot\varphi + \partial\partial\cdot\varphi' + \partial^2\varphi'' , \quad (4.4)$$

so that eq. (4.3) is equivalent to

$$\mathcal{F}' - \partial^2\varphi'' = 3\square\partial\alpha + 2\partial^2\partial\cdot\alpha . \quad (4.5)$$

On the other hand, the trace of eq. (4.2) is

$$\mathcal{F}' = 3\square\partial\alpha + 6\partial^2\partial\cdot\alpha + 3\partial^3\alpha' , \quad (4.6)$$

and thus, by comparison, one obtains

$$\partial^2\varphi'' = 4\partial^2\partial\cdot\alpha + \partial^3\alpha' = \partial^2(4\partial\cdot\alpha + \partial\alpha') . \quad (4.7)$$

The conclusion is that the triplet equations imply a pair of *local* equations for a single massless spin- $s$  gauge field  $\varphi$  and a single spin- $(s - 3)$  compensator  $\alpha$ ,

$$\begin{aligned} \mathcal{F} &= 3\partial^3\alpha , \\ \varphi'' &= 4\partial\cdot\alpha + \partial\alpha' , \end{aligned} \quad (4.8)$$

that are invariant under the *unconstrained* gauge transformations

$$\delta\varphi = \partial\Lambda , \quad (4.9)$$

$$\delta\alpha = \Lambda' , \quad (4.10)$$

and clearly reduce to the standard Fronsdal form after a partial gauge fixing using the trace  $\Lambda'$  of the gauge parameter. These equations are nicely consistent, since the second is

implied by the first, as can be seen using the Bianchi identity of eq. (2.34). However, *these are not Lagrangian equations*, somewhat in the spirit of the Vasiliev form of higher-spin dynamics [9, 10].

#### 4.2. Non-Lagrangian formulation in (A)dS

One can also obtain the (A)dS extension of the spin- $s$  compensator equations (4.8). To this end, the starting point are the (A)dS gauge transformations for the fields  $\varphi$  and  $\alpha$ , that in such a curved background take naturally the form

$$\begin{aligned}\delta \varphi &= \nabla \Lambda , \\ \delta \alpha &= \Lambda' .\end{aligned}\tag{4.11}$$

One can then proceed in various ways, for instance starting from the gauge variation of the (A)dS form of the Fronsdal operator

$$\begin{aligned}\delta \mathcal{F}_L &\equiv \delta \left\{ \mathcal{F} - \frac{1}{L^2} [(3 - \mathcal{D} - s)(2 - s) - s]\varphi - 2g\varphi' \right\} \\ &= 3(\nabla^3 \Lambda') - \frac{4}{L^2} g \nabla \Lambda' ,\end{aligned}\tag{4.12}$$

and it is then simple to conclude that the compensator form of the higher-spin equations in (A)dS is

$$\begin{aligned}\mathcal{F} &= 3\nabla^3 \alpha + \frac{1}{L^2} \{[(3 - \mathcal{D} - s)(2 - s) - s]\varphi + 2g\varphi'\} - \frac{4}{L^2} g \nabla \alpha , \\ \varphi'' &= 4\nabla \cdot \alpha + \nabla \alpha' .\end{aligned}\tag{4.13}$$

These are again nicely consistent: making use of the Bianchi identity of eq. (3.10) one can in fact verify that the first of (4.13) implies the second. However, Lagrangian equations can be obtained, both in flat space and in an (A)dS background, from a BRST construction based on a wider set of constraints, an issue to which we now turn.

#### 4.3. BRST analysis and compensator Lagrangian in flat space

The previous constructions show that the BRST machinery encodes quite neatly the physical state conditions one wants to describe for these systems, and on the other hand provides a direct path toward their inclusion in suitable off-shell formulations. Thus for

the flat-space triplet one builds the BRST operator, as in [15], out of the three generators  $(\ell_0, \ell_{\pm 1})$ , and this leads to a description where the field  $\varphi$  is eventually subject to the conditions

$$\square \varphi = 0, \quad \partial \cdot \varphi = 0, \quad (4.14)$$

that indeed propagate a chain of modes of spins  $s, s-2, \dots, 0$  or  $1$  according to whether  $s$  is even or odd. In a similar fashion, the description of irreducible spin- $s$  modes would require the additional on-shell constraint

$$\varphi' = 0, \quad (4.15)$$

and this would bring about the operators  $M$  and  $M^\dagger$  that we have already met in subsection 3.2. However, while there we treated them as structure constants of the  $(\ell_0, \ell_{\pm 1})$  triplet algebra, here we shall introduce corresponding ghost-antighost pairs  $(c_{\pm M}, b_{\pm M})$ , that as usual satisfy the anti-commutation relations

$$\{ c_{\pm 1M}, b_{\mp 1M} \} = 1, \quad (4.16)$$

and resort to the construction of [17], whose result is indeed an off-shell system that embodies the compensator equations (4.8). A more complicated BRST construction, described in [24], adapted to the non-linear constraint algebra (3.23) and (3.24), would also determine the AdS deformation of this system, that we shall not discuss for brevity.

Let us therefore begin by reviewing the results in [17], whose BRST procedure rests on the algebra

$$\begin{aligned} [\ell_1, M^\dagger] &= \ell_{-1}, & [\ell_{-1}, M] &= -\ell_1, & [\ell_1, \ell_{-1}] &= \ell_0, \\ [M, M^\dagger] &= N, & [M, N] &= 2M, & [M^\dagger, N] &= -2M^\dagger, \end{aligned} \quad (4.17)$$

where  $N$  and  $M$  are defined in eqs. (3.21) and (3.22). Notice that the new operators,  $M$ ,  $M^\dagger$  and  $N$ , close on an  $SO(1,2)$  subalgebra. The BRST construction for this system presents an interesting subtlety, since  $N$ , a *strictly positive* operator, cannot be regarded as providing a physical state condition in the spirit of (4.14). Hence, although the algebra is formally closed, it effectively includes second-class constraints associated with  $M$  and  $M^\dagger$ , whose commutator gives rise to the offending operator  $N$ . A way out, however, is provided in [17, 24], whose basic idea is to eliminate the offending constraint via an

auxiliary realization of the algebra involving an additional oscillator,  $d$ , that we shall take to satisfy the commutation relation

$$[d, d^\dagger] = -1 , \quad (4.18)$$

and an additional parameter,  $h$ , that plays the role of the non-trivial dynamical value of the offending constraint. In practice, one can dispose of this constraint altogether, rotating it away by a suitable unitary transformation built from the conjugate momentum of  $h$ . The states in the enlarged Fock space are expanded, as usual, in (anti)ghost modes, and each of the resulting terms,

$$|\varphi_i\rangle = \sum_k |\varphi_{i,k}\rangle \equiv \sum_k \varphi_{i;\mu_1\mu_2\dots\mu_p}^k \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_p} (d^\dagger)^k |0\rangle , \quad (4.19)$$

comprises arbitrary powers of the new  $d^\dagger$  oscillator. Although this expansion is formally an infinite series, as we shall see the number of powers of  $d^\dagger$  needed to describe a spin- $s$  field is actually finite.

More specifically, in order to eliminate the offending constraint  $N$ , one first modifies it, including in it a parameter  $h$ , to be regarded as an additional phase-space variable. If this were done naively, however, the algebra (4.17) would not be preserved. The way out is precisely to introduce the additional degrees of freedom associated to the oscillator  $d$  and to build an auxiliary realization for the algebra. Clearly this complication is not needed for  $\ell_0$  and  $\ell_{\pm 1}$ , that are first-class constraints. On the other hand,  $M^\pm$  and  $N$  close on an SO(2,1) subalgebra, for which a convenient recipe is available, precisely in terms of the single new oscillator  $d$  of eq. (4.18). It is in fact simple to verify that

$$\begin{aligned} M_{(aux)} &= d \sqrt{h + 1 + d^\dagger d} , \\ M_{(aux)}^\dagger &= d^\dagger \sqrt{h + d^\dagger d} , \\ N_{(aux)} &= -2 d^\dagger d - h \end{aligned} \quad (4.20)$$

close on the SO(2,1) algebra (4.17). Since they clearly commute with the original  $M$  and  $N$  operators, one can define new operators,

$$\tilde{M}_\pm = M^\pm + M_{(aux)}^\pm , \quad \tilde{N} = N + N_{(aux)} , \quad (4.21)$$

that realize again the SO(2,1) algebra (4.17). The nilpotent BRST charge for the resulting system is then formally constructed, treating all operators under consideration as first class constraints, as

$$\begin{aligned} Q = & c_0 \ell_0 + c_1 \ell_{-1} + c_M \tilde{M}^\dagger + c_{-1} \ell_1 + c_{-M} \tilde{M} + c_N \tilde{N} \\ & - c_{-1} c_1 b_0 + c_{-1} b_{-1} c_M - c_{-M} c_1 b_1 \\ & + c_N (2c_{-N} b_N + 2b_{-N} c_N + c_{-1} b_1 + b_{-1} c_1 - 3) - c_{-M} c_M b_N . \end{aligned} \quad (4.22)$$

The final step is the elimination of the term proportional to  $c_N$  while maintaining the nilpotency of the BRST charge. This can be done performing on the BRST charge the unitary transformation

$$Q \rightarrow e^{-i\pi x_h} Q e^{i\pi x_h} , \quad (4.23)$$

where  $x_h$  is the phase-space coordinate conjugate to  $h$ , so that

$$[x_h, h] = i , \quad (4.24)$$

and

$$\pi = M - 2d^\dagger d + 2c_{-N} b_N + 2b_{-N} c_N + c_{-1} b_1 + b_{-1} c_1 - 3 \quad (4.25)$$

is essentially a number operator. Notice that this transformation removes all terms depending on  $c_N$  from the BRST charge, while obviously preserving its nilpotency. Finally, the term containing  $b_N$  can be also dropped without any effect on the nilpotency, and one is left with a BRST charge without the offending constraint, but where the other constraints are suitably redefined by (4.23).

Therefore, after the unitary transformation that rotates away the offending constraint, the *identically nilpotent* BRST charge for this system takes the form

$$Q = Q_1 + Q_2 , \quad (4.26)$$

with

$$\{Q_1, Q_2\} = 0 , \quad Q_1^2 = -Q_2^2 , \quad (4.27)$$

where

$$\begin{aligned} Q_1 = & c_0 \ell_0 + c_1 \ell_{-1} + c_M M^\dagger + c_{-1} \ell_1 + c_{-M} M \\ & - c_{-1} c_1 b_0 + c_{-1} b_{-1} c_M - c_{-M} c_1 b_1 , \end{aligned} \quad (4.28)$$

and

$$\begin{aligned} Q_2 = & c_{-M} \sqrt{-1 + N - d^\dagger d + 2 b_{-M} c_M + 2 c_{-M} b_M + b_{-1} c_1 + c_{-1} b_1} d \\ & + d^\dagger \sqrt{-1 + N - d^\dagger d + 2 b_{-M} c_M + 2 c_{-M} b_M + b_{-1} c_1 + c_{-1} b_1} c_M. \end{aligned} \quad (4.29)$$

Again, this determines a BRST invariant Lagrangian of the type (2.31), and now the most general expansions of the state vector  $|\Phi\rangle$  and of the gauge parameter  $|\Lambda\rangle$  in ghost variables are

$$\begin{aligned} |\Phi\rangle = & |\varphi_1\rangle + c_{-1} b_{-1} |\varphi_2\rangle + c_{-M} b_{-M} |\varphi_3\rangle + c_{-1} b_{-M} |\varphi_4\rangle \\ & + c_{-M} b_{-1} |\varphi_5\rangle + c_{-1} c_{-M} b_{-1} b_{-M} |\varphi_6\rangle + c_0 b_{-1} |C_1\rangle \\ & + c_0 b_{-M} |C_2\rangle + c_0 c_{-1} b_{-1} b_{-M} |C_3\rangle + c_0 c_{-M} b_{-1} b_{-M} |C_4\rangle, \end{aligned} \quad (4.30)$$

and

$$\begin{aligned} |\Lambda\rangle = & b_{-1} |\Lambda_1\rangle + b_{-M} |\Lambda_2\rangle + c_{-1} b_{-1} b_{-M} |\Lambda_3\rangle + c_{-M} b_{-1} b_{-M} |\Lambda_4\rangle \\ & + c_0 b_{-1} b_{-M} |\Lambda_5\rangle, \end{aligned} \quad (4.31)$$

where  $|\varphi_i\rangle$  and  $|C_i\rangle$  have ghost number  $g = -1/2$  and depend *only* on the bosonic creation operators  $\alpha_{-1}^\mu$  and  $d^\dagger$ . Let us also note that both the Lagrangian and the gauge transformations are not affected by redefinitions of the gauge parameters of the type

$$\delta |\Lambda\rangle = Q |\omega\rangle, \quad (4.32)$$

and in particular with

$$|\omega\rangle = b_{-1} b_{-M} |\omega_1\rangle. \quad (4.33)$$

As a result, one of the gauge parameters,  $|\Lambda_5\rangle$ , is inessential and can be ignored.

With this proviso, the resulting Lagrangian in the bosonic Fock-space notation is

$$\begin{aligned} \mathcal{L} = & -\langle C_1|C_1\rangle - \langle C_2|\varphi_2\rangle + \langle C_3|\varphi_3\rangle + \langle C_4|C_4\rangle - \langle \varphi_2|C_2\rangle + \langle \varphi_3|C_3\rangle \\ & - \langle C_1|M^\dagger|\varphi_4\rangle - \langle C_1|\ell_{-1}|\varphi_2\rangle + \langle C_1|\ell_1|\varphi_1\rangle - \langle C_2|M^\dagger|\varphi_3\rangle - \langle C_2|\ell_{-1}|\varphi_5\rangle \\ & + \langle C_2|M|\varphi_1\rangle - \langle C_3|M^\dagger|\varphi_6\rangle + \langle C_3|\ell_1|\varphi_5\rangle - \langle C_3|M|\varphi_2\rangle + \langle C_4|\ell_{-1}|\varphi_6\rangle \\ & + \langle C_4|\ell_1|\varphi_3\rangle - \langle C_4|M|\varphi_4\rangle + \langle \varphi_1|M^\dagger|C_2\rangle + \langle \varphi_1|\ell_{-1}|C_1\rangle - \langle \varphi_1|\ell_0|\varphi_1\rangle \end{aligned}$$

$$\begin{aligned}
& - \langle \varphi_2 | M^\dagger | C_3 \rangle + \langle \varphi_2 | \ell_0 | \varphi_2 \rangle - \langle \varphi_2 | \ell_1 | C_1 \rangle + \langle \varphi_3 | \ell_{-1} | C_4 \rangle + \langle \varphi_3 | \ell_0 | \varphi_3 \rangle \\
& - \langle \varphi_3 | M | C_2 \rangle - \langle \varphi_4 | M^\dagger | C_4 \rangle + \langle \varphi_4 | \ell_0 | \varphi_5 \rangle - \langle \varphi_4 | M | C_1 \rangle - \langle \varphi_5 | \ell_{-1} | C_3 \rangle \\
& + \langle \varphi_5 | \ell_0 | \varphi_4 \rangle - \langle \varphi_5 | \ell_1 | C_2 \rangle - \langle \varphi_6 | \ell_0 | \varphi_6 \rangle + \langle \varphi_6 | \ell_1 | C_4 \rangle - \langle \varphi_6 | M | C_3 \rangle \\
& - \langle C_1 | d^\dagger X_1 | \varphi_4 \rangle - \langle C_2 | d^\dagger X_2 | \varphi_3 \rangle + \langle C_2 | X_0 d | \varphi_1 \rangle - \langle C_3 | d^\dagger X_{1,4} | \varphi_6 \rangle \\
& - \langle C_3 | X_2 d | \varphi_2 \rangle - \langle C_4 | X_3 d | \varphi_4 \rangle + \langle \varphi_1 | d^\dagger X_0 | C_2 \rangle - \langle \varphi_2 | d^\dagger X_2 | C_3 \rangle \\
& - \langle \varphi_3 | X_2 d | C_2 \rangle - \langle \varphi_4 | d^\dagger X_3 | C_4 \rangle - \langle \varphi_4 | X_1 d | C_1 \rangle - \langle \varphi_6 | X_4 d | C_3 \rangle , \quad (4.34)
\end{aligned}$$

where

$$X_r = \sqrt{-1 + N - d^\dagger d + r} . \quad (4.35)$$

In the compact index-free tensorial notation, the same Lagrangian reads

$$\begin{aligned}
\mathcal{L} = & \sum_k \left[ Y_{k,0} \varphi_1^k \square \varphi_1^k - Y_{k,2} \varphi_2^k \square \varphi_2^k - Y_{k,4} \varphi_3^k \square \varphi_3^k - Y_{k,3} \varphi_4^k \square \varphi_5^k - Y_{k,3} \varphi_5^k \square \varphi_4^k \right. \\
& + Y_{k,6} \varphi_6^k \square \varphi_6^k - Y_{k,1} (C_1^k)^2 + 2 Y_{k,1} C_2^k \varphi_2^k - 2 Y_{k,4} C_3^k \varphi_3^k + Y_{k,5} (C_4^k)^2 \\
& - Y_{k,3} (C_1^k)' \varphi_4^k - 2 Y_{k,1} C_1^k \partial \varphi_2^k - 2 Y_{k,0} C_1^k \partial \varphi_1^k + Y_{k,4} (C_2^k)' \varphi_3^k \\
& - 2 Y_{k,2} C_2^k \partial \varphi_5^k - Y_{k,2} C_2^k (\varphi_1^k)' + Y_{k,6} (C_3^k)' \varphi_6^k - 2 Y_{k,3} \varphi_5^k \partial C_3^k \\
& + Y_{k,4} C_3^k (\varphi_2^k)' + 2 Y_{k,5} C_4^k \partial \varphi_6^k - Y_{k,4} \varphi_3^k \partial C_4^k - 2 Y_{k,5} C_4^k (\varphi_4^k)' \\
& \left. - 2 \sqrt{s-k-3 + \frac{\mathcal{D}}{2}} (-Y_{k,3} C_1^{k+1} \varphi_4^k + Y_{k,4} C_2^{k+1} \varphi_3^k - Y_{k,2} C_2^k \varphi_1^{k+1} \right. \\
& \left. + Y_{k,6} C_3^{k+1} \varphi_6^k + Y_{k,4} C_3^k \varphi_2^{k+1} + Y_{k,5} C_4^k \varphi_4^{k+1}) \right] , \quad (4.36)
\end{aligned}$$

with

$$Y_{k,r} = \frac{(-1)^k}{(s-2k-r)!} . \quad (4.37)$$

The complete field equations are then

$$\begin{aligned}
& -\eta C_2^k - \partial C_1^k + \square \varphi_1^k - k \sqrt{s-k-2 + \frac{\mathcal{D}}{2}} C_2^{k-1} = 0 , \\
& C_2^k + \eta C_3^k - \square \varphi_2^k + \partial \cdot C_1^k + k \sqrt{s-k-2 + \frac{\mathcal{D}}{2}} C_3^{k-1} = 0 , \\
& -C_3^k - \partial C_4^k - \square \varphi_3^k + \frac{1}{2}(C_2^k)' - \sqrt{s-k-3 + \frac{\mathcal{D}}{2}} C_2^{k+1} = 0 ,
\end{aligned}$$

$$\begin{aligned}
& \eta C_4^k + \square \varphi_5^k + \frac{1}{2} (C_1^k)' + k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} C_4^{k-1} - \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} C_1^{k+1} = 0 , \\
& \partial C_3^k + \square \varphi_4^k - \partial \cdot C_2^k = 0 , \\
& \square \varphi_6^k - \partial \cdot C_4^k + \frac{1}{2} (C_3^{k+1})' - \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} C_3^{k+1} = 0 , \\
& C_1^k + \eta \varphi_4^k + \partial \varphi_2^k - \partial \cdot \varphi_1^k - k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \varphi_4^{k-1} = 0 , \\
& \eta \varphi_3^k - \partial \varphi_5^k - \frac{1}{2} (\varphi_1^k)' + k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \varphi_3^{k-1} - \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \varphi_1^{k+1} + \varphi_2^k = 0 , \\
& \eta \varphi_6^k + \partial \cdot \varphi_5^k + \frac{1}{2} (\varphi_2^k)' + k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \varphi_6^{k-1} - \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \varphi_2^{k+1} - \varphi_3^k = 0 , \\
& C_4^k - \partial \varphi_6^k + \partial \cdot \varphi_3^k - \frac{1}{2} (\varphi_4^k)' + \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \varphi_4^{k+1} = 0 , \tag{4.38}
\end{aligned}$$

where  $\eta$  denotes the Minkowski metric, while the corresponding gauge transformations are

$$\begin{aligned}
& \delta \varphi_1^k = \partial \Lambda_1^k + \eta \Lambda_2^k + k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \Lambda_2^{k-1} , \\
& \delta \varphi_2^k = \Lambda_2^k + \partial \cdot \Lambda_1^k + \eta \Lambda_3^k + k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \Lambda_3^{k-1} , \\
& \delta \varphi_3^k = -\Lambda_3^k + \frac{1}{2} (\Lambda_2^k)' - \partial \Lambda_4^k - \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \Lambda_2^{k+1} , \\
& \delta \varphi_4^k = \partial \cdot \Lambda_2^k - \partial \Lambda_3^k , \\
& \delta \varphi_5^k = -\eta \Lambda_4^k - \frac{1}{2} (\Lambda_1^k)' - k \sqrt{s - k - 2 + \frac{\mathcal{D}}{2}} \Lambda_4^{k-1} + \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \Lambda_1^{k+1} , \\
& \delta \varphi_6^k = -\frac{1}{2} (\Lambda_3^k)' + \partial \cdot \Lambda_4^k + \sqrt{s - k - 3 + \frac{\mathcal{D}}{2}} \Lambda_3^{k+1} , \\
& \delta C_1^k = \square \Lambda_1^k , \\
& \delta C_2^k = \square \Lambda_2^k , \\
& \delta C_3^k = \square \Lambda_3^k , \\
& \delta C_4^k = \square \Lambda_4^k . \tag{4.40}
\end{aligned}$$

From the field equations and the gauge transformations one can unambiguously read the oscillator content of the vectors  $|\varphi_i\rangle$ ,  $|C_i\rangle$  and  $|\Lambda_i\rangle$ . In order to describe a spin- $s$  field,

let us fix the number of oscillators  $\alpha_{-1}^\mu$  in the zeroth-order term of the expansion of  $|\varphi_1\rangle$  in the oscillator  $d^\dagger$ , that we shall denote by  $\varphi_1^0$ , to be equal to  $s$ . This is actually the field  $\varphi$  of the previous subsections, while all other terms describe auxiliary or compensator fields. The zeroth-order components in the  $d^\dagger$  oscillators for the other fields have thus the following  $\alpha_{-1}^\mu$  content, here summarized in terms of the resulting total spin, displayed within brackets:  $\varphi_2^0 [s-2]$ ,  $\varphi_3^0 [s-4]$ ,  $\varphi_4^0 [s-3]$ ,  $\varphi_5^0 [s-3]$ ,  $\varphi_6^0 [s-6]$ ,  $C_1^0 [s-1]$ ,  $C_2^0 [s-2]$ ,  $C_3^0 [s-4]$ ,  $C_4^0 [s-5]$ ,  $\Lambda_1^0 [s-1]$ ,  $\Lambda_2^0 [s-2]$ ,  $\Lambda_3^0 [s-4]$ ,  $\Lambda_4^0 [s-5]$ . Moreover, the field equations and the gauge transformations show that each power of the  $d^\dagger$  oscillator reduces the number of  $\alpha_{-1}^\mu$  oscillators by two units, so that, for instance, the  $\varphi_1^k$  component field has  $s-2k$  oscillators of this type, and thus spin  $(s-2k)$ . Therefore, as anticipated, in this off-shell formulation a spin- $s$  field requires finitely many auxiliary fields and gauge transformation parameters, although their total number grows linearly with  $s$ .

Combining the gauge transformations with the field equations, it is possible to choose a gauge where all fields aside from  $\varphi_1^0, \varphi_2^0, \varphi_5^0$  and  $C_1^0$  are eliminated, so that one is left with a reduced set of equations invariant under an unconstrained gauge symmetry of parameter  $\Lambda_1^0$ . To this end, one first gauges away all fields  $C_i^k$  but  $C_1^0$ , and the residual gauge transformations are restricted by the conditions

$$\ell_0 \Lambda_1^k = 0 \quad (k \neq 0) \quad \text{and} \quad \ell_0 \Lambda_i^k = 0 \quad (i = 2, 3, 4) \quad \text{and} \quad k \geq 0 . \quad (4.41)$$

The parameters  $\Lambda_1^k$  ( $k \neq 0$ ) and  $\Lambda_4^k$  gauge away  $\varphi_5^k$  ( $k \neq 0$ ), while the parameters  $\Lambda_3^k$  gauge away  $\varphi_6^k$ . The conclusion is that one is finally left with gauge transformation parameters restricted by the additional condition

$$(M + X_4 d) |\Lambda_3\rangle = 0 , \quad (4.42)$$

and with the help of these parameters  $|\Lambda_2\rangle$  and  $|\Lambda_3\rangle$  one can also gauge away  $\varphi_1^k, \varphi_2^k$  ( $k \neq 0$ ) and  $\varphi_3^k$ , while  $\varphi_4^k$  vanishes as a result of the field equations.

One can now identify  $\varphi_1^0$  with  $\varphi$ ,  $\varphi_2^0$  with  $D$ ,  $C_1^0$  with  $C$ ,  $-2 \varphi_5^0$  with the compensator  $\alpha$  and  $\Lambda_1^0$  with the gauge parameter of the previous subsections. The first, second, seventh and eighth equations in (4.38) then produce the triplet and compensator equations of the previous subsections, while the fourth and ninth equations are consequences of these.

This construction is clearly somewhat complicated with respect to the non-Lagrangian equations (4.8). For instance, the off-shell description of a spin-4 field  $\varphi_{\mu\nu\rho\sigma}$  makes use

of thirteen different fields, out of which, however, eight are of  $D$  type and five are of  $C$  type, that can be simply eliminated. One can use the additional gauge parameters and field equations to eliminate all fields aside from the original  $\varphi$ , its two triplet partners and the compensator, whose relation to the triplet fields is now one of the residual equations of motion rather than a constraint as in subsection 4.1. For brevity, we refrain from discussing the (A)dS extension of these results, that is similarly related to the analysis in [24]. The far simpler triplets, as we have seen, provide an alternative description of irreducible higher-spin multiplets in (A)dS backgrounds.

## 5. The fermionic triplets

We can now turn to the fermionic triplets, that were proposed in [4] as a natural guess for the field equations of symmetric spinor-tensors arising in the tensionless limit of superstrings. As we shall see, they indeed emerge in this limit, although, as is usually the case for the fermionic string, the dominant types of fields are (generalized) forms rather than symmetric tensors. Actually, we shall not be able to pursue the analysis to the same level of detail as in the previous sections. Thus, while we shall derive both triplet and Vasiliev-like compensator equations for higher-spin fermionic gauge fields, we shall not be able to present a corresponding compensator Lagrangian formulation, since it is being constructed by other authors using the same BRST approach discussed in the previous section [32]. Moreover, we shall not be able to extend the fermionic triplets to off-shell systems in (A)dS, and here the BRST analysis will explain clearly the difficulty, related to the nature of the algebra of the resulting deformed constraints.

### 5.1. Open superstring oscillators

Most of the results of the previous sections can be naturally extended to superstrings. For brevity, we restrict our attention to the open sector of the type-I superstring, but closed superstrings could be treated in a similar way. Let us first perform the  $\alpha' \rightarrow \infty$  limit in the BRST charge for the open superstring

$$Q = \sum_{-\infty}^{+\infty} \left[ L_{-n} C_n + G_{-r} \Gamma_r - \frac{1}{2} (m-n) : C_{-m} C_{-n} B_{m+n} : \right]$$

$$+ \left( \frac{3n}{2} + m \right) : C_{-n} \mathcal{B}_{-m} \Gamma_{m+n} : - \Gamma_{-n} \Gamma_{-m} \mathcal{B}_{m+n} \Big] - a C_0 , \quad (5.1)$$

where  $a$  is the intercept and the super-Virasoro generators

$$\begin{aligned} L_k &= \frac{1}{2} \sum_{l=-\infty}^{+\infty} \alpha_{k-l} \alpha_l + \frac{1}{4} \sum_r (2r - l) \psi_{l-r} \psi_r , \\ G_r &= \sum_{l=-\infty}^{+\infty} \alpha_l \psi_{r-l} , \end{aligned} \quad (5.2)$$

obey the super-Virasoro algebra

$$\begin{aligned} [L_k, L_l] &= (k - l) L_{k+l} + \frac{\mathcal{D}}{8} (k^3 - k) , \\ [L_k, G_r] &= \left( \frac{k}{2} - r \right) G_{k+r} , \\ \{G_r, G_s\} &= 2 L_{r+s} + \frac{\mathcal{D}}{2} \left( r^2 - \frac{1}{4} \right) \delta_{rs} . \end{aligned} \quad (5.3)$$

Here  $(k, l)$  are integers for both the Neveu-Schwarz (NS) and Ramond (R) sectors, while  $(r, s)$  are integers for the R sector and half-odd integers for the NS sector,  $\mathcal{D}$  denotes once more the space-time dimension ( $\mathcal{D} = 10$  for the tensile string) and  $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$ . The fermionic oscillators  $\psi_r^\mu$  and the ghosts  $\Gamma_r$  and antighosts  $\mathcal{B}_r$  satisfy

$$\{\psi_r^\mu, \psi_s^\nu\} = \delta_{r+s,0} \eta^{\mu\nu} , \quad [\Gamma_r, \mathcal{B}_s] = i \delta_{r+s,0} , \quad (5.4)$$

and the intercept is  $a = 0$  in the R sector and  $a = \frac{1}{2}$  in the NS sector.

Rescaling the ghost variables as

$$\gamma_{-r} = \sqrt{2\alpha'} \Gamma_{-r} , \quad \beta_r = \frac{1}{\sqrt{2\alpha'}} \mathcal{B}_r \quad (5.5)$$

and then taking the  $\alpha' \rightarrow \infty$  limit, one obtains the nilpotent BRST charge for the NS sector

$$Q_{NS} = c_0 \ell_0 + \tilde{Q}_{NS} - M_{NS} b_0 , \quad (5.6)$$

with

$$\begin{aligned} \tilde{Q}_{NS} &= \sum_{k \neq 0} [c_{-k} \ell_k + \gamma_{-r} g_r] , \\ M_{NS} &= \frac{1}{2} \sum_{-\infty}^{+\infty} [k c_{-k} c_k + \gamma_{-r} \gamma_r] , \end{aligned} \quad (5.7)$$

and

$$g_r = p \cdot \psi_r . \quad (5.8)$$

In a similar fashion, the limiting BRST charge for the R sector reads

$$Q_R = c_0 \ell_0 + \gamma_0 g_0 + \tilde{Q}_R - M_R b_0 - \frac{1}{2} \gamma_0^2 b_0 , \quad (5.9)$$

where  $\tilde{Q}_R$  and  $M_R$  are again given by (5.7), the only difference being that their sums are over half-odd integer modes for fermionic Virasoro generators and bosonic (anti)ghosts. Both BRST charges are again identically nilpotent, independently of the space-time dimension  $\mathcal{D}$ .

For the type I superstring, the string field is invariant under the action of the BRST invariant GSO projection operators

$$P_{NS} = \frac{1}{2} \left[ 1 - (-1)^{\psi_p^\dagger \psi_p + i \gamma_p^\dagger \beta_p - i \gamma_p \beta_p^\dagger} \right] \quad (5.10)$$

and

$$P_R = \frac{1}{2} \left[ 1 + \gamma_{11} (-1)^{\psi_r^\dagger \psi_r + i \gamma_r^\dagger \beta_r - i \gamma_r \beta_r^\dagger + i \gamma_0 \beta_0} \right] , \quad (5.11)$$

where  $\gamma_{11}$  is the ten-dimensional chirality matrix, that apply to the NS and R sectors respectively. Expanding the NS string field and the gauge parameter in terms of the fermionic ghost zero mode as

$$\begin{aligned} |\Phi^{NS}\rangle &= |\Phi_1^{NS}\rangle + c_0 |\Phi_2^{NS}\rangle , \\ |\Lambda^{NS}\rangle &= |\Lambda_1^{NS}\rangle + c_0 |\Lambda_2^{NS}\rangle , \end{aligned} \quad (5.12)$$

and making use of the BRST charge (5.6), one obtains the field equations

$$\begin{aligned} \ell_0 |\Phi_1^{NS}\rangle - \tilde{Q}_{NS} |\Phi_2^{NS}\rangle &= 0 , \\ \tilde{Q}_{NS} |\Phi_1^{NS}\rangle - M_{NS} |\Phi_2^{NS}\rangle &= 0 , \end{aligned} \quad (5.13)$$

along with the gauge transformations

$$\begin{aligned} \delta |\Phi_1^{NS}\rangle &= \tilde{Q}_{NS} |\Lambda_1^{NS}\rangle - M_{NS} |\Lambda_2^{NS}\rangle , \\ \delta |\Phi_2^{NS}\rangle &= \ell_0 |\Lambda_1^{NS}\rangle - \tilde{Q}_{NS} |\Lambda_2^{NS}\rangle . \end{aligned} \quad (5.14)$$

The R sector is more complicated, due to the presence of the bosonic ghost zero mode  $\gamma_0$ . However, one can work with the truncated string field

$$|\Phi^R\rangle = |\Phi_1^R\rangle + \gamma_0 |\Phi_2^R\rangle + 2c_0 g_0 |\Phi_2^R\rangle , \quad (5.15)$$

while still preserving the relevant portion of the gauge symmetry and, of course, not affecting the physical spectrum [37]. The resulting, consistently truncated, field equations

$$\begin{aligned} g_0 |\Phi_1^R\rangle + \tilde{Q}_R |\Phi_2^R\rangle &= 0 , \\ \tilde{Q}_R |\Phi_1^R\rangle - 2M_R g_0 |\Phi_2^R\rangle &= 0 , \end{aligned} \quad (5.16)$$

are then invariant under the gauge transformations

$$\begin{aligned} \delta |\Phi_1^R\rangle &= \tilde{Q}_R |\Lambda_1^R\rangle + 2M_R g_0 |\Lambda_2^R\rangle , \\ \delta |\Phi_2^R\rangle &= g_0 |\Lambda_1^R\rangle - \tilde{Q}_R |\Lambda_2^R\rangle . \end{aligned} \quad (5.17)$$

### 5.2. Symmetric spinor-tensors

If, as for the bosonic string, one considers fields  $|\Phi^{R,1}\rangle$  and  $|\Phi^{R,2}\rangle$  depending only on the bosonic oscillator  $\alpha_{-1}^\mu$  and on the fermionic ghost variables  $c_{-1}$  and  $b_{-1}$ , the expansions

$$\begin{aligned} |\Phi_1^R\rangle &= \frac{1}{n!} \psi_{\mu_1\mu_2\dots\mu_n}(x) \alpha_{-1}^{\mu_1} \alpha_{-1}^\mu \dots \alpha_{-1}^{\mu_n} |0\rangle \\ &\quad + \frac{1}{(n-2)!} \lambda_{\mu_1\mu_2\dots\mu_{n-2}}(x) \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_{n-2}} |0\rangle , \\ |\Phi_2^R\rangle &= - \frac{1}{\sqrt{2} (n-1)!} \chi_{\mu_1\mu_2\dots\mu_{n-1}}(x) \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_{n-1}} |0\rangle \end{aligned} \quad (5.18)$$

define spinor-tensor fields  $\psi$ ,  $\chi$  and  $\lambda$  totally symmetric in their tensor indices and of spin  $(n+1/2)$ ,  $(n-1/2)$  and  $(n-3/2)$ , respectively. Substituting these expressions in the field equations (5.16) - (5.16) then yields precisely the fermionic triplet equations of [4]:

$$\begin{aligned} \partial\psi &= \partial\chi , \\ \partial\cdot\psi - \partial\lambda &= \partial\chi , \\ \partial\lambda &= \partial\cdot\chi . \end{aligned} \quad (5.19)$$

The BRST gauge invariance involves an unconstrained parameter,

$$|\Lambda'_1\rangle = \frac{1}{(n-1)!} \epsilon_{\mu_1\mu_2\dots\mu_{n-1}}(x) \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_{n-1}} |0\rangle , \quad (5.20)$$

and determines the gauge transformations

$$\begin{aligned} \delta\psi &= \partial\epsilon , \\ \delta\Lambda &= \partial\cdot\epsilon , \\ \delta\chi &= \partial\epsilon , \end{aligned} \quad (5.21)$$

in agreement with [4].

Let us note, however, that the totally symmetric bosonic triplets of subsection 2.3 do not arise directly in the NS sector of the open superstring, since all states containing only bosonic  $\alpha^\mu$  oscillators and fermionic  $b, c$  ghosts are eliminated by the GSO projection operator (5.10). However, they can emerge from tensors with mixed symmetry, or even directly if the GSO projection is modified to correspond to type-0 strings [38]. Generalized triplets of mixed symmetry are actually the superpartners of symmetric fermionic triplets in the type-I superstring.

One can also consider generalized triplets for spinor-tensors, that also arise in the R sector, and these, described by

$$\begin{aligned} g_0 |\lambda_{i_1\dots i_l}^{j_1\dots j_l}\rangle - (-1)^l \ell_{i_l} |\lambda_{i_1\dots i_{l-1}}^{j_1\dots j_l}\rangle + (-1)^l \ell_{-j} |\lambda_{i_1\dots i_l}^{j j_1\dots j_{l-1}}\rangle &= 0 , \\ \ell_{i_l} |\lambda_{i_1\dots i_{l-1}}^{j_1\dots j_{l-1}}\rangle - \ell_{-j} |\lambda_{i_1\dots i_l}^{j j_1\dots j_{l-1}}\rangle - 2g_0 (-1)^l |\lambda_{i_1\dots i_{l-1}}^{i_l j_1\dots j_{l-1}}\rangle &= 0 , \end{aligned} \quad (5.22)$$

resemble the generalized bosonic triplets of subsection 2.4.

These equations follow from the Lagrangians

$$\begin{aligned} \mathcal{L} &= \sum_l \left[ \frac{(-1)^l}{(l!)^2} \langle \lambda_{j_1\dots j_l}^{i_1\dots i_l} | g_0 |\lambda_{i_1\dots i_l}^{j_1\dots j_l}\rangle - \frac{2}{(l-1)!l!} \langle \lambda_{i_1\dots i_{l-1}}^{j_1\dots j_l} | \ell_{-i_l} |\lambda_{j_1\dots j_l}^{i_1\dots i_l}\rangle \right. \\ &\quad \left. - \frac{2(-1)^l}{(l!)^2} \langle \lambda_{i_1\dots i_l}^{j_1\dots j_{l+1}} | \ell_{j_{l+1}} |\lambda_{j_1\dots j_l}^{i_1\dots i_l}\rangle + \frac{(-1)^l}{((l-1)!)^2} \langle \lambda_{j_1\dots j_{l-1}}^{i_1\dots i_{l-1} j_l} | g_0 |\lambda_{i_1\dots i_{l-1}}^{j_1\dots j_l}\rangle \right] . \end{aligned} \quad (5.23)$$

that are invariant under the gauge transformations

$$\delta |\lambda_{i_1\dots i_l}^{j_1\dots j_l}\rangle = -(-1)^l \ell_{i_l} |\Lambda_{i_1\dots i_{l-1}}^{1(1) j_1\dots j_l}\rangle + (-1)^l \ell_{-j} |\Lambda_{i_1\dots i_l}^{1(1) j, j_1\dots j_l}\rangle$$

$$\begin{aligned}
 & + 2 g_0 |\Lambda_{i_1, \dots, i_{l-1}}^{2(1) i_l, j_1, \dots, j_l} \rangle , \\
 \delta |\chi_{i_1, \dots, i_{l-1}}^{j_1, \dots, j_l} \rangle & = g_0 |\Lambda_{i_1, \dots, i_{l-1}}^{1(1) j_1, \dots, j_l} \rangle - (-1)^l \ell_{i_{l-1}} |\Lambda_{i_1, \dots, i_{l-2}}^{2(1) j_1, \dots, j_l} \rangle \\
 & + (-1)^l \ell_{-j} |\Lambda_{i_1, \dots, i_{l-1}}^{(1) j, j_1, \dots, j_l} \rangle ,
 \end{aligned} \tag{5.24}$$

that also allow the “gauge-for-gauge” transformations

$$\begin{aligned}
 \delta |\Lambda_{i_1, \dots, i_l}^{1(k) j_1, \dots, j_{l+k}} \rangle & = -(-1)^l \ell_{i_l} |\Lambda_{i_1, \dots, i_{l-1}}^{1(k+1) j_1, \dots, j_{l+k}} \rangle + (-1)^l \ell_{-j} |\Lambda_{i_1, \dots, i_l}^{1(k+1) j, j_1, \dots, j_{l+k}} \rangle \\
 & + 2 |\Lambda_{i_1, \dots, i_{l-1}}^{2(k+1) i_l, j_1, \dots, j_{l+k}} \rangle , \\
 \delta |\Lambda_{i_1, \dots, i_{l-1}}^{2(k) j_1, \dots, j_{l+k}} \rangle & = g_0 |\Lambda_{i_1, \dots, i_{l-1}}^{1(k+1) j_1, \dots, j_{l+k}} \rangle - (-1)^l \ell_{i_{l-1}} |\Lambda_{i_1, \dots, i_{l-2}}^{2(k+1) j_1, \dots, j_{l+k}} \rangle \\
 & + (-1)^l \ell_{-j} |\Lambda_{i_1, \dots, i_{l-1}}^{2(k+1) j, j_1, \dots, j_{l+k}} \rangle ,
 \end{aligned} \tag{5.25}$$

and so on.

The “mixed symmetry” of these fields is of general type, and allowing for the possible dependence of the string field on  $\psi_{-r}$  and  $\gamma_{-r}, \beta_{-r}$  in the R sector would lead to more complicated equations with similar properties.

### 5.3. Compensator form of the fermionic higher-spin equations

Here the story parallels the discussion in subsection 4.1, since the fermionic Fang-Fronsdal operator [2]

$$\mathcal{S} = i (\not{\partial} \psi - \partial \not{\psi}) \tag{5.26}$$

varies into a term proportional to the gamma-trace of the gauge parameter,

$$\delta \mathcal{S} = -2i \partial^2 \not{\epsilon}, \tag{5.27}$$

under the gauge transformation

$$\delta \psi = \partial \epsilon. \tag{5.28}$$

In addition,  $\mathcal{S}$  satisfies the Bianchi identity

$$\partial \cdot \mathcal{S} - \frac{1}{2} \partial \mathcal{S}' - \frac{1}{2} \not{\partial} \not{\mathcal{S}} = i \partial^2 \not{\psi}', \tag{5.29}$$

and as a result the gauge parameter and the gauge field were constrained in [2] to satisfy the conditions

$$\not{\epsilon} = 0, \quad \not{\psi}' = 0. \tag{5.30}$$

As for integer-spin fields, one can eliminate these constraints either passing to the non-local equations of [3] or, alternatively, introducing a single compensator field  $\xi$ . The resulting equations,

$$\begin{aligned}\mathcal{S} &= -2i\partial^2\xi, \\ \psi' &= 2\partial\cdot\xi + \partial\xi' + \not{\partial}\not{\xi},\end{aligned}\tag{5.31}$$

are then invariant under the gauge transformations

$$\begin{aligned}\delta\psi &= \partial\epsilon, \\ \delta\xi &= \not{\epsilon},\end{aligned}\tag{5.32}$$

involving an unconstrained gauge parameter, and are consistent, since the first implies the second via the Bianchi identity (5.29).

These compensator equations generalize nicely to an (A)dS background. The gauge transformation for a spin- $s$  fermion becomes in this case

$$\delta\psi = \nabla\epsilon + \frac{1}{2L}\gamma\epsilon,\tag{5.33}$$

where, as in previous sections,  $\nabla$  denotes an (A)dS covariant derivative and  $L$  determines the (A)dS curvature. In order to proceed, one needs the commutator of two covariant derivatives on a spin-1/2 field  $\eta$ ,

$$[\nabla_\mu, \nabla_\nu]\eta = -\frac{1}{2L^2}\gamma_{\mu\nu}\eta,\tag{5.34}$$

where  $\gamma_{\mu\nu}$  is antisymmetric in  $\mu$  and  $\nu$  and equals the product  $\gamma_\mu\gamma_\nu$  when  $\mu$  and  $\nu$  are different, that can be combined with eq. (3.4) to obtain the corresponding expression for fields of arbitrary half-odd integer spins.

For a spin- $s$  fermion ( $s = n + \frac{1}{2}$ ), where  $n$  is the number of vector indices carried by the field  $\psi$ , the compensator equations in an (A)dS background are

$$\begin{aligned}(\not{\nabla}\psi - \nabla\psi) + \frac{1}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{1}{2L}\gamma\psi \\ = -\{\nabla, \nabla\}\xi + \frac{1}{L}\gamma\nabla\xi + \frac{3}{2L^2}g\xi, \\ \psi' = 2\nabla\cdot\xi + \not{\nabla}\not{\xi} + \nabla\xi' + \frac{1}{2L}[\mathcal{D} + 2(n-2)]\not{\xi} - \frac{1}{2L}\gamma\xi',\end{aligned}\tag{5.35}$$

and are invariant under

$$\begin{aligned}\delta\psi &= \nabla\epsilon, \\ \delta\xi &= \not{\epsilon},\end{aligned}\tag{5.36}$$

with an unconstrained parameter  $\epsilon$ . Eqs. (5.35) are again a pair of non-Lagrangian equations, like their flat-space counterparts (5.31), and are again nicely consistent, on account of the (A)dS deformation of the Bianchi identity (5.29),

$$\begin{aligned}\nabla\cdot\mathcal{S} - \frac{1}{2}\nabla\mathcal{S}' - \frac{1}{2}\not{\nabla}\not{\mathcal{S}} &= \frac{i}{4L}\gamma S' + \frac{i}{4L}[(\mathcal{D}-2) + 2(n-1)]\not{\mathcal{S}} \\ &+ \frac{i}{2}\left[\{\nabla,\nabla\} - \frac{1}{L}\gamma\nabla - \frac{3}{2L^2}\right]\not{\psi}',\end{aligned}\tag{5.37}$$

where now the Fang-Fronsdal operator  $\mathcal{S}$  is also deformed and becomes

$$\mathcal{S} = i(\not{\nabla}\psi - \nabla\psi) + \frac{i}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{i}{2L}\gamma\psi.\tag{5.38}$$

We have been unable to construct a corresponding Lagrangian AdS deformation for generic fermionic triplets. Already for the simplest case of a  $(3/2, 1/2)$  system, that involves a pair of fields  $\psi_\mu$  and  $\chi$ , one can write the (A)dS equations

$$\begin{aligned}\not{\nabla}\psi_\mu + \frac{\mathcal{D}-2}{2L}\psi_\mu + \frac{1}{2L}\gamma_\mu\psi &= \nabla_\mu\chi, \\ \nabla\cdot\psi + \frac{\mathcal{D}-1}{2L}\psi &= \not{\nabla}\chi\end{aligned}\tag{5.39}$$

that are invariant under the gauge transformations

$$\begin{aligned}\delta\psi_\mu &= \nabla_\mu\epsilon + \frac{1}{2L}\gamma_\mu\epsilon \\ \delta\chi &= \not{\nabla}\epsilon + \frac{\mathcal{D}}{2L}\epsilon,\end{aligned}\tag{5.40}$$

but, when suitably combined, they give rise to the further condition

$$\chi = \psi.\tag{5.41}$$

As for the bosonic triplet, the modes described by this system thus reduce to a single spin multiplet, but differently from that case the additional constraints do not arise in the gauge-fixing procedure, but are generated by the field equations themselves. The origin of

these difficulties is clearly spelled by the BRST analysis. In this context, a key problem one is facing is that, after adapting the Dirac operator  $g_0 = \gamma \cdot p$  to the (A)dS background, the resulting set of operators  $\ell_0$ ,  $\ell_{\pm 1}$  and  $g_0$  does not form a closed algebra, not even a non-linear one as was the case for integer-spin fields on AdS. The way out would be to enlarge the constraint algebra, including in it the additional operators  $T^\pm = \gamma \cdot \alpha^\pm$  corresponding to gamma-trace conditions, as was done for the  $M^\pm$  operators corresponding to ordinary traces in subsection 3.2, and then to construct a nilpotent BRST charge at expense of the inclusion of further ghost fields as in [24]. While we hope to return to this point in the near future, the additional constraints would lead to an off-shell description of an irreducible spin multiplet not directly related to the triplet structures we were after in this work.

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### Note added

The original version of this paper contained an improper interpretation of the consistency conditions for the gauge-fixing of (A)dS triplets, that we interpreted as signaling that the spectrum would collapse to irreducible spin- $s$  modes. In fact, the problem was just the reflection of an unsuitable gauge choice. We are very grateful to the referee for calling to our attention the difficulties related with our interpretation.

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